CHAPTER 5

Polynomials

5.0. Foreword

Polynomials of various kinds are used extensively in numerical analysis. This chapter presents some general aspects of polynomials. First some comments are made regarding polynomials in general. These are followed by discussions of the more specific topics of Lagrange and Hermite polynomials.

5.1. General Polynomials

5.1.1. Polynomials in One Independent Variable

In one dimension a complete *n*-th degree polynomial is given by

$$P_n(x) = \sum_{i=0}^{n} \alpha_i x^i \tag{5.1}$$

Clearly the number of terms in the polynomial is equal to $(n + 1)$. The constants α_i are sometimes referred to as *generalized coordinates*.

Example 5.1: some polynomials in one dimension

- For $n = 1$:

$$P_1(x) = \alpha_0 + \alpha_1 x \qquad \text{(2 terms)}$$

- For $n = 3$:

$$P_3(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \qquad \text{(4 terms)}$$

5.1.2. Polynomials in Two Independent Variables

In two dimensions a complete *n*-th degree polynomial is given by

$$P_n(x,y) = \sum_{k=0}^{n} \alpha_k x^i y^j \qquad i + j \le k \tag{5.2}$$

where i and j are permuted accordingly. The number of terms in the above polynomial is equal to $(n+1)(n+2)/2$.

Example 5.2: some polynomials in two dimensions

- For $n = 1$:

$$P_1(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y \qquad \text{(3 terms)}$$

- For $n = 3$:

$$P_3(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 x^2$$
$$+\alpha_6 x^3 + \alpha_7 x^2 y + \alpha_8 xy^2 + \alpha_9 y^3 \qquad \text{(10 terms)}$$

A convenient way in which to illustrate the terms which make up a two-dimensional polynomial is through the Pascal triangle shown in Figure 5.1.

Name								*No. of terms*
constant				1				1
linear			x		y			3
quadratic			x^2	xy	y^2			6
cubic		x^3	x^2y		xy^2	y^3		10
quartic		x^4	x^3y	x^2y^2	xy^3	y^4		15
quintic	x^5	x^4y	x^3y^2		x^2y^3	xy^4	y5	21
hexadic	x^6	x^5y	x^4y^2	x^3y^3	x^2y^4	xy^5	y^6	28

Figure 5.1. Pascal Triangle Associated with Two-Dimensional Polynomials

5.1.3. Polynomials in Three Independent Variables

In three dimensions a complete n-th degree polynomial is given by

$$P_n(x,y,z) = \sum_{m=0}^{n} \alpha_m x^i y^j z^k \qquad i + j + k \leq m \tag{5.3}$$

where i, j and k are permuted accordingly. The number of terms in the above polynomial is equal to $(n + 1)(n + 2)(n + 3)/6$.

Example 5.3: some polynomials in three dimensions

- For $n = 1$:

$$P_1(x,y,z) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z \qquad \text{(4 terms)}$$

- For $n = 2$:

$$P_2(x,y,z) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 xy + \alpha_5 xz + \alpha_6 yz + \alpha_7 x^2 + \alpha_8 y^2 + \alpha_9 z^2 \qquad \text{(10 terms)}$$

A convenient way in which to illustrate the terms which make up a three-dimensional polynomial is through a tetrahedron which is a three-dimensional extension of the Pascal triangle shown in Figure 5.1.

5.2. Lagrange Polynomials

Theorem

The Lagrange interpolating polynomial of *f(x)*, denoted by $L_n(x)$, is given by ($L_n(x)$ is of degree $\leq n$)

$$L_n(x) = \sum_{i=0}^{n} f(x_i) \prod_{k=0,\, k\neq i}^{n} \frac{(x - x_k)}{(x_i - x_k)}$$

$$= \sum_{i=0}^{n} f(x_i) \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} \tag{1}$$

NOTE: this type of polynomial is computationally intensive for large values of *n*.

proof: (heuristic)

In order for $L_n(x)$ to be an interpolating polynomial, we must show that $L_n(x_j) = f(x_j)$. From equation (1) it follows that

$$L_n(x_j) = \sum_{i=0}^{n} f(x_i) \frac{(x_j - x_0)(x_j - x_1)\cdots(x_j - x_{i-1})(x_j - x_{i+1})\cdots(x_j - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} \tag{2}$$

If $j \neq i$, it follows that since $j = k$ is possible, that one of the quantities in parentheses in the numerator will be zero for each of these *n* terms in the summation. Finally, if $j = i$, then clearly

$$L_n(x_j) = f(x_i)(1) = f(x_j) \tag{3}$$

which is exactly the desired results. Thus, $L_n(x)$ is an interpolating polynomial.

5.3. Osculating (Hermite) Polynomials

Consider a case in which an analyst desires an interpolating polynomial that not only agrees with the function values at a discrete number of points, but whose derivative(s) also agree(s) with the derivative(s) of the function at these same points. These needs can be fulfilled by so-called *osculating*[1] polynomials.

The set of osculating polynomials is a generalization of both the Taylor polynomials discussed in Chapter 3 and the Lagrange polynomials discussed in Section 5.2 above [1]. Osculating polynomials have the property that, given $(n + 1)$ distinct points $x_0, x_1, x_2, \ldots, x_n$ and non-negative integers $m_0, m_1, m_2, \ldots, m_n$, the osculating polynomial approximating a function $f \in C^m[a,b]$, where $m = \max\{m_0, m_1, m_2, \ldots, m_n\}$ and $x_i \in [a,b]$ for each $i = 0, \ldots, n$, is the polynomial of least degree with the property that it agrees with the function f and all of its derivatives of order less than or equal to m_i at the point x_i for each $i = 0, 1, \ldots, n$. The degree of this osculating polynomial will be at most

$$M = \sum_{i=0}^{n} m_i + n \tag{5.4}$$

since the number of conditions to be satisfied is $\sum_{i=0}^{n} m_i + (n+1)$, and a polynomial of degree M has $M+1$ coefficients that can be used to satisfy these conditions. A formal definition of an osculating polynomial is given below [1].

Definition : osculating polynomials

Let $x_0, x_1, \ldots, x_n$ be $(n + 1)$ distinct points in $[a,b]$ and mi be a non-negative integer associated with x_i for $i = 0, 1, \ldots, n$. Let

$$m = \max_{0 \le i \le n} m_i \tag{5.5}$$

and $f \in C^m[a,b]$. The osculating polynomial approximating f is the polynomial P of least degree such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k} \tag{5.6}$$

for each $i = 0, 1, \ldots, n$ and $k = 0, 1, \ldots, m_i$.

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[1] to osculate means to touch two curves at three or more points.

When $n = 0$, the osculating polynomial approximating f is simply the Taylor polynomial of degree m_0 for f at x_0 (see Chapter 3). When $m_i = 0$ for $i = 0, 1, \ldots, n$, the osculating polynomial is simply the Lagrange polynomial interpolating f on $x_0, x_1, \ldots, x_n$.

If $m_i = 1$ for each $i = 0, 1, \ldots, n$, the resulting class of polynomials is referred to as *Hermite polynomials*. For a given function f, these polynomials not only agree with f at the points $x_0, x_1, \ldots, x_n$, but, since their first derivatives agree with those of f, they possess the same "shape" as f at these points (i.e., the tangent lines to the polynomial and to the function agree at these points).

Theorem : Hermite polynomial for $m = 1 = m_i$

If $f \in C^1[a,b]$ and $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct, the unique polynomial of least degree agreeing with f and its first derivative at $x_0, x_1, \ldots, x_n$ is given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x) \tag{5.7}$$

where

$$H_{n,j}(x) = \left[1 - 2(x - x_j) L'_{n,j}(x_j)\right] L^2_{n,j}(x) \tag{5.8}$$

$$\hat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x) \tag{5.9}$$

and

$$L_{n,j}(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)} \tag{5.10}$$

Equation (5.10) is simply the j-th Lagrange coefficient polynomial of degree n defined in Section 5.2. The proof of this theorem is found in reference [1].

In formulating finite elements, Hermite polynomials are commonly used as interpolation functions for Bernoulli-Euler beam elements.

5.4. References

1. R. L. Burden, J. D. Faires and A. C. Reynolds. Numerical Analysis. PWS Publishing, Boston, MA, 1980.