Notes Concerning Matrices and Linear Algebra

Contents

0. Introduction

The practical use of numerical methods is based on matrix algebra and on the use of the digital computer. It is only in matrix form that the complete solution process can be expressed in an elegant yet compact manner. The objective of this appendix is to present a brief overview of the fundamentals of matrices and linear algebra which are required for an understanding of solution procedures associated with numerical methods. Further details concerning matrices and linear algebra can be found in any of a number of standard books on the subject [1-3].

- **Definition :** A matrix is a rectangular or linear assemblage of numbers or variables arranged in rows and in columns. The numbers or variables comprising a matrix are called *elements* or *entries* of the matrix. The elements may be real or they may be complex.

- **Notation :** A matrix will be denoted by an uppercase Latin letter either as $[A]$, $\mathbf{A}$ or, when written on a medium not supporting bold characters, as $\underset{\sim}{A}$.

- **Notation :** An element of a matrix will be denoted by a lowercase Latin letter with two subscripts; e.g., the elements of $\mathbf{A}$ are denoted by a_{ij}, where i represents the *row number* and j the *column number*.

- **Notation :** On occasion it is desirable to show the size of a matrix along side its symbol. Thus to explicitly indicate the dimensions of the (m x n) matrix $\mathbf{A}$, we write $\mathbf{A}_{(m\,x\,n)}$, $[A]_{(m\,x\,n)}$ or $\underset{\sim}{A}_{(m\,x\,n)}$.

- **Definition :** An n-vector is a collection of n real or complex numbers $v_1, v_2, \cdots, v_n$ arranged in a column; e.g.,

$$\{v\} = \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ \vdots \\ v_n \end{Bmatrix}$$

- **Notation :** Vectors will be denoted by lowercase Latin symbols as $\{v\}$, $\mathbf{v}$, or when written on a medium not supporting bold characters, as $\underset{\sim}{v}$.

1. Special Matrices

• **Definition :** Column Vectors

The (m x 1) matrix **c** which has the form

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_{m1} \end{bmatrix}$$

Since no distinction is made between (m x 1) matrices and m-vectors, such a matrix is called an m-dimensional *column vector*. Like general vectors, column vectors will be denoted by lowercase Latin symbols as $\{c\}$, $\mathbf{c}$, or $\underset{\sim}{c}$.

• **Definition :** Row Vectors

The (1 x n) matrix **r** has the form $\mathbf{r} = \langle r_{11} \quad r_{12} \quad \cdots \quad \cdots \quad r_{1n} \rangle$

Such a matrix is called an n-dimensional *row vector*. Like general vectors, row vectors will be denoted by lowercase Latin symbols as $\langle r \rangle$, $\mathbf{r}$, or $\underset{\sim}{r}$.

• **Definition :** Square Matrices

An (m x n) matrix $\mathbf{A}$ is *square* if $m = n$.

• **Definition :** Diagonal / Superdiagonal / Subdiagonal

Let $\mathbf{A}$ be an (m x n) matrix and let $k = \min(m, n)$.

The elements a_{ii} (i = 1, 2, ... , k) are said to lie on the *diagonal* of $\mathbf{A}$.

The elements $a_{i,i+1}$ are said to lie on the *superdiagonal* of $\mathbf{A}$.

The elements $a_{i,i-1}$ are said to lie on the *subdiagonal* of $\mathbf{A}$.

EXAMPLE 1 :

Consider the matrix
$$\mathbf{A} = \begin{bmatrix} \delta & \alpha & a_{13} & a_{14} & a_{15} \\ \beta & \delta & \alpha & a_{24} & a_{25} \\ a_{31} & \beta & \delta & \alpha & a_{35} \\ a_{41} & a_{42} & \beta & \delta & a_{45} \end{bmatrix}$$

The *diagonal* elements are denoted by δ; the *superdiagonal* elements are denoted by α; and, the *subdiagonal* elements are denoted by β.

♦

• **Definition :** Diagonal Matrices

A square matrix is *diagonal* if its only nonzero elements lie on the diagonal.

EXAMPLE 2 :

An example of a diagonal matrix is given below

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} = diag\left(a_{11} \quad a_{22} \quad a_{33} \quad a_{44}\right)$$

If all the nonzero diagonal elements equal to 1, $\mathbf{A} = \mathbf{I}$ = the *identity matrix*.

♦

• **Definition :** Upper Trapezoidal Matrices

An (m x n) matrix $\mathbf{A}$ is *upper trapezoidal* if $a_{ij} = 0$ for $i > j$.

• **Definition :** Lower Trapezoidal Matrices

An (m x n) matrix $\mathbf{A}$ is *lower trapezoidal* if $a_{ij} = 0$ for $i < j$.

EXAMPLE 3 :

The matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \end{bmatrix}$ is upper trapezoidal.

♦

• **Definition :** Upper (Lower) Triangular Matrices

A *square* upper (lower) trapezoidal matrix is said to be *upper* (*lower*) *triangular*. Some special kinds of triangular matrices: If $\mathbf{T}$ is upper (lower) triangular with *zero* diagonal elements, then $\mathbf{T}$ is said to be *strictly upper* (*lower*) *triangular*. If the diagonal elements of $\mathbf{T}$ are unity, then $\mathbf{T}$ is said to be *unit upper* (*lower*) *triangular*.

EXAMPLE 4 :

The matrix $\mathbf{L} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ is lower triangular.

The matrix $\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ t_{21} & 0 & 0 & 0 \\ t_{31} & t_{32} & 0 & 0 \\ t_{41} & t_{42} & t_{43} & 0 \end{bmatrix}$ is strictly lower triangular.

The matrix $\mathbf{U} = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is *unit* upper triangular.

♦

Remark : From the above it follows that a matrix is *diagonal* if and only if it is *both* upper and lower triangular.

Remark : The identity matrix is *both* unit upper and unit lower triangular.

• **Definition :** Hessenberg Matrices

A square matrix $\mathbf{H}$ is *upper Hessenberg* if $i > j+1 \Rightarrow h_{ij} = 0$; It is *lower Hessenberg* if $i < j-1 \Rightarrow h_{ij} = 0$.

EXAMPLE 5 :

The matrix $\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & 0 & 0 & 0 \\ h_{21} & h_{22} & h_{23} & 0 & 0 \\ h_{31} & h_{32} & h_{33} & h_{34} & 0 \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} \\ h_{51} & h_{52} & h_{53} & h_{54} & h_{55} \end{bmatrix}$ is (5 x 5) lower Hessenberg.

♦

• **Definition :** Band Matrices ; Bandwidth

Consider a square matrix $\mathbf{A}$. If $|i-j|>k \quad \Rightarrow a_{ij}=0$, then $\mathbf{A}$ is called a *band matrix*. The *bandwidth* for a matrix of this type is equal to $2k+1$. The *half-bandwidth* is equal to $k+1$.

EXAMPLE 6 :

Consider the square matrix $$\mathbf{A}=\begin{bmatrix} 2 & 1 & 3 & 0 & 0 & 0 \\ 1 & 4 & 3 & 1 & 0 & 0 \\ 3 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & 8 & 2 & 1 & 6 \\ 0 & 0 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & 9 & 2 & 6 \end{bmatrix}$$

From row 3 (i.e., $i=3$) it is evident that $a_{ij}=0$ for $j>5$ and $j<1$, implying that $|i-j|=|3-5|=2$ or $|i-j|=|3-1|=2$. Consequently, $k=2$, and the *bandwidth* is equal to $2(2)+1=5$.

♦

Remark : for k = 0, 1, 2 $\mathbf{A}$ is referred to as being *diagonal*, *tridiagonal*, and *pentadiagonal*, respectively.

EXAMPLE 7 :

A tridiagonal matrix has its non-zero elements arranged in a band along the diagonal, with a bandwidth equal to 3. For example,

$$\mathbf{A}=\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

♦

Remark : If it is *both* upper and lower Hessenberg, a square matrix is *tridiagonal*.

• **Definition :** Symmetric Matrices

A square matrix $\mathbf{A}$ of order n is symmetric if $a_{ij} = a_{ji} \quad ; \quad i, j = 1, 2, \cdots, n$.

Remark : The tridiagonal matrix shown in EXAMPLE 7 is symmetric.

• **Definition :** Strictly Diagonally Dominant Matrices

An (n x n) matrix $\mathbf{A}$ is said to be *strictly diagonally dominant* if

$$|a_{ii}| > \sum_{j=1; j \neq i}^{n} |a_{ij}|$$

holds for each $i = 1, 2, \cdots, n$.

NOTE: An (n x n) matrix $\mathbf{A}$ is said to be *diagonally dominant* if

$$|a_{ii}| \geq \sum_{j=1; j \neq i}^{n} |a_{ij}|$$

holds for each $i = 1, 2, \cdots, n$.

• **Definition :** Positive Definite Matrices

A symmetric (n x n) matrix $\mathbf{A}$ is called *positive definite* if the relation $\mathbf{x}^T \mathbf{A} \mathbf{x} > \mathbf{0}$ holds for every n-dimensional vector $\mathbf{x}$.

NOTE: a weaker case is when $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq \mathbf{0}$ holds for every n-dimensional vector $\mathbf{x}$. In this case is said to be *positive semi-definite*.

2. Algebraic Operations on Matrices

• **Definition :** Equality of Matrices

Two matrices **A** and **B** are said to be equal if and only if:

1) Both are of the same size (e.g., m x n); and,

2) $a_{ij} = b_{ij}$ for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$.

• **Definition :** Addition of Two Matrices

Two matrices can be added if and only if they have the same dimensions. Thus let **A** and **B** be (m x n) matrices. The *sum* of **A** and **B** (i.e., $\mathbf{C} = \mathbf{A} + \mathbf{B}$) is the matrix **C** whose elements are given by:

$$c_{ij} = a_{ij} + b_{ij}$$

where $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$.

• Programming Note : The process of matrix addition defined above is easily translated into the following sequence of FORTRAN statements (note the correspondence between indices):

```
      subroutine matadd (m,n,nr,nc,a,b,c)
*
*   add the matrices A and B, placing the sum in C.
*    nr = number of active rows in a matrix
*    nc = number of active columns in a matrix
*
      integer i,j,m,n,nc,nr
      real a(m,n),b(m,n),c(m,n)
*
      do 200 i=1,nr
          do 100 j=1,nc
              c(i,j) = a(i,j) + b(i,j)
  100     continue
  200 continue
      return
      end
```

♦

• **Theorem :** Properties of Matrix Addition

Let **A**, **B** and **C** be (m x n) matrices. Then

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutative property)
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (associative property)

• **Definition :** Multiplication of Matrices by Scalars

Let $\mathbf{A}$ be an $(m \times n)$ matrix and let λ be a real number (i.e., a scalar).

- The product of λ and $\mathbf{A}$ is written $\lambda\mathbf{A}$.
- The elements of $\lambda\mathbf{A}$ are given by λa_{ij}, where $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$; i.e., each element of $\mathbf{A}$ is multiplied by the scalar λ.

• **Theorem :** Scalar Multiplication of Matrices

Let $\mathbf{A}$ and $\mathbf{B}$ be matrices and let λ and μ be real numbers. Then

- $(\lambda\mu)\mathbf{A} = \lambda(\mu\mathbf{A})$
- $(\lambda + \mu)\mathbf{A} = \lambda\mathbf{A} + \mu\mathbf{A}$
- $\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$
- $(1)\mathbf{A} = \mathbf{A}$

NOTE: matrix *subtraction* involves the addition of a matrix multiplied by the scalar (-1); i.e.,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$$

• **Definition :** The Zero Matrix

Any matrix whose elements are all zero is called a *zero matrix* and is denoted by the symbol $[0]$, $\mathbf{0}$ or $\underset{\sim}{0}$.

- for any matrix $\mathbf{A}$:

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$

and

$$\mathbf{A} + (-1)\mathbf{A} = \mathbf{A} + (-\mathbf{A}) = \mathbf{0}$$

• **Definition :** Product of Two Matrices

Let $\mathbf{A}$ be an $(l \times m)$ matrix and let $\mathbf{B}$ be an $(m \times n)$ matrix. The product of $\mathbf{A}$ and $\mathbf{B}$ is the $(l \times n)$ matrix $\mathbf{P}$; i.e., $\mathbf{P} = \mathbf{AB}$. The elements of $\mathbf{P}$ are given by:

$$p_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

where $i = 1, 2, \cdots, l$ and $j = 1, 2, \cdots, n$. As evident from the above equation, matrix multiplication consists of the scalar (dot) product of row vectors in the *pre-multiplier* $\mathbf{A}$ and column vectors in the *post-multiplier* $\mathbf{B}$.

NOTES:

- The product of two matrices is defined only if the matrices are conformable; i.e., the *prefactor* matrix (or *pre-multiplier*) has the same number of columns as the *postfactor* matrix (or *post-multiplier*) has rows.

- The product of two matrices generalizes the notion of multiplication of two vectors; i.e., it is the result of forming a matrix whose elements are scalar (dot) products obtained by multiplying row vectors in the *prefactor* with column vectors in the *postfactor* .

EXAMPLE 8 :

Let **A** be a (2 x 3) matrix and let **B** be a (3 x 3) matrix. Their product is then given by

$$\mathbf{P}_{2\,x\,3} = \mathbf{A}_{2\,x\,3}\mathbf{B}_{3\,x\,3}$$
$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where

$$p_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$
$$p_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$
$$p_{13} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}$$
$$p_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$
$$p_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$$
$$p_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$$

♦

• Programming Note : The process of matrix multiplication defined by

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj} \quad ; \quad i = 1, 2, \cdots, l \,;\, j = 1, 2, \cdots, n$$

is easily translated into the following sequence of FORTRAN 77 statements (note the correspondence between indices):

```
      subroutine matmul (l,m,n,nr,nc,nrc,a,b,c)
*
*  compute the matrix product of A and B, storing the result in C.
*   nr = number of rows active in [C]
*   nc = number of columns active in [C]
*   nrc = number of active rows in [B] or columns in [A]
*
     integer i,j,k,l,m,n,nc,nr,nrc
     real a(l,m), b(m,n), c(l,n)
*
     do 300  i=1,nr
         do 200 j=1,nc
             c(i,j) = 0.0
```

```
            do 100 k=1,nrc
               c(i,j) = c(i,j) + a(i,k)*b(k,j)
100         continue
200      continue
300 continue
    return
    end
```

♦

• **Theorem** : Properties Associated with Multiplication of Matrices

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be matrices and let λ be a real number. Then

- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- $\mathbf{A}(\mathbf{B} \pm \mathbf{C}) = \mathbf{AB} \pm \mathbf{AC}$
- $(\mathbf{A} \pm \mathbf{B})\mathbf{C} = \mathbf{AC} \pm \mathbf{BC}$
- $\lambda(\mathbf{AB}) = (\lambda\mathbf{A})\mathbf{B} = \mathbf{A}(\lambda\mathbf{B})$

NOTES:

• In general, matrix multiplication is *not* commutative; i.e., even if $\mathbf{A}$ and $\mathbf{B}$ have conformable dimensions, $\mathbf{AB} \neq \mathbf{BA}$.

EXAMPLE 9 :

Let $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$. It follows that

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 13 \\ 8 & 20 \end{bmatrix}$$

and

$$\mathbf{BA} = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 24 & 2 \\ 10 & -1 \end{bmatrix}$$

♦

EXAMPLE 10 :

Consider the following matrix product:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This shows that it is possible for the product of two non-zero matrices to be zero. From this result it follows that if $\mathbf{AX} = \mathbf{AY}$ with $\mathbf{A} \neq \mathbf{0}$, we cannot conclude that $\mathbf{X} = \mathbf{Y}$. That is, *the cancellation law for real numbers does not extend to matrices.*

♦

• **Theorem** : Pre- or Post- Multiplication of Matrices by the Identity Matrix

Let $\mathbf{A}$ be an (m x n) matrix, and let $\mathbf{I}$ be the identity matrix. Then $\mathbf{I}_m\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}$.

NOTE: Powers of Matrices.

Let $\mathbf{A}$ be a square matrix and let n be a positive integer. Then

$$\mathbf{A}^n = \mathbf{A}\mathbf{A}\cdots\cdots\mathbf{A} \quad ; \quad \text{i.e., } n \text{ factors}$$

Having defined the process of matrix multiplication, it follows that the set of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

can be written in the more compact form $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Furthermore, matrix multiplication, when applied to two systems of linear equations, greatly simplifies manipulations associated with their solution. For example, given one system of linear equations which express several x variables in terms of y variables, i.e.,

$$\begin{aligned} a_{11}y_1 + a_{12}y_2 &= x_1 \\ a_{21}y_1 + a_{22}y_2 &= x_2 \\ a_{31}y_1 + a_{32}y_2 &= x_3 \end{aligned}$$

and a second set of linear equations which express y variables in terms of z variables:

$$\begin{aligned} y_1 &= c_{11}z_1 + c_{12}z_2 \\ y_2 &= c_{21}z_1 + c_{22}z_2 \end{aligned}$$

In matrix form it follows that

$$\mathbf{A}_{(3\,x\,2)}\,\mathbf{y}_{(2\,x\,1)} = \mathbf{x}_{(3\,x\,1)}$$

and

$$\mathbf{y}_{(2\,x\,1)} = \mathbf{C}_{(2\,x\,2)}\,\mathbf{z}_{(2\,x\,1)}$$

To obtain new equations which express the x variables in terms of the z variables, simply substitute the second equation above into the first, giving

$$\mathbf{A}_{(3x2)}\,\mathbf{y}_{(2x1)} = \mathbf{A}_{(3x2)}\mathbf{C}_{(2x2)}\,\mathbf{z}_{(2x1)} = \mathbf{x}_{(3x1)}$$

or upon expansion

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

• **Theorem** : Products of Special Matrices

The product of a diagonal matrix and a *diagonal* (*trapezoidal*, *Hessenberg*, *tridiagonal*) matrix is *diagonal* (*trapezoidal*, *Hessenberg*, *tridiagonal*). The product of two upper (lower) *trapezoidal* matrices is upper (lower) *trapezoidal*.

3. Matrix Operations

• **Definition :** Matrix Transposition

Let $\mathbf{A}$ be an (m x n) matrix. The transpose of $\mathbf{A}$, denoted by $\mathbf{A}^T$, is the (m x n) matrix obtained by writing the rows of $\mathbf{A}$ as the columns of $\mathbf{A}^T$ and the columns of $\mathbf{A}$ as the rows of $\mathbf{A}^T$; i.e., $a_{ij}^T = a_{ji}$.

EXAMPLE 11 :

The transpose of a column vector is a row vector. Thus, the transpose of $\mathbf{b} = \begin{Bmatrix} 5 \\ 1 \\ 2 \\ 0 \\ -6 \end{Bmatrix}$ is

$$\mathbf{b}^T = \{5 \quad 1 \quad 2 \quad 0 \quad -6\}.$$

♦

EXAMPLE 12 :

The transpose of a matrix is obtained by simply interchanging its rows and columns; e.g.,

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 5 & -5 \\ 1 & 0 & 3 & 2 \\ 5 & 2 & 6 & 0 \end{bmatrix} \quad ; \quad \mathbf{A}^T = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 0 & 2 \\ 5 & 3 & 6 \\ -5 & 2 & 0 \end{bmatrix}$$

♦

• **Comment :** A matrix is *symmetric* if and only if it is equal to its transpose; i.e., if $\mathbf{A} = \mathbf{A}^T$ or $a_{ij} = a_{ji}$.

• **Comment :** If $\mathbf{A}^T = -\mathbf{A}$; i.e., if $a_{ij} = -a_{ji}$, the matrix is said to be *skew-symmetric* (or *anti-symmetric*). In a skew-symmetric matrix all the diagonal elements are equal to *zero*.

• **Theorem :** Operations Associated with Matrix Transposition

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be matrices and let λ be a real number. Then

$$\left(\mathbf{A}^T\right)^T = \mathbf{A}$$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$
$$(\lambda\mathbf{A})^T = \lambda\mathbf{A}^T$$
$$(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$$

NOTE: the fourth relation above holds for any number of matrices.

EXAMPLE 13 :

For any matrix $\mathbf{A}$, the matrices $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are defined and are symmetric. To prove this assertion, first note that the matrix sizes are conformable for multiplication; i.e., the number of columns in $\mathbf{A}$ is, by definition, equal to the number of rows in $\mathbf{A}^T$, and the number of columns in $\mathbf{A}^T$ is equal to the number of rows in $\mathbf{A}$. Next consider the following manipulations:

$$\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\left(\mathbf{A}^T\right)^T = \mathbf{A}^T\mathbf{A} \qquad QED$$

♦

EXAMPLE 14 :

Let $\mathbf{x}$ and $\mathbf{y}$ be n-dimensional vectors. Then the product $\mathbf{x}^T\mathbf{y}$ is a *scalar* whose value is

$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

This scalar is called the *inner product* of $\mathbf{x}$ and $\mathbf{y}$.

♦

• **Definition :** Decomposition of an Unsymmetric Matrix (Euclidean decomposition of a second-rank tensor)

Let $\mathbf{D}$ be a square unsymmetric matrix. It may be decomposed into *symmetric* and *skew-symmetric* parts; *viz*.,

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{D}+\mathbf{D}^T\right) + \frac{1}{2}\left(\mathbf{D}-\mathbf{D}^T\right)$$

or

$$d_{ij} = \frac{1}{2}\left(d_{ij}+d_{ji}\right) + \frac{1}{2}\left(d_{ij}-d_{ji}\right)$$

To show that the matrix sum $\frac{1}{2}\left(\mathbf{D}+\mathbf{D}^T\right)$ is symmetric, note that

$$\left(\frac{1}{2}\left(\mathbf{D}+\mathbf{D}^T\right)\right)^T = \frac{1}{2}\left(\mathbf{D}^T+\mathbf{D}\right) = \frac{1}{2}\left(\mathbf{D}+\mathbf{D}^T\right) \qquad QED$$

To show that the matrix difference $\frac{1}{2}\left(\mathbf{D}-\mathbf{D}^T\right)$ is skew-symmetric, note that

$$\left(\frac{1}{2}\left(\mathbf{D}-\mathbf{D}^T\right)\right)^T=\frac{1}{2}\left(\mathbf{D}^T-\mathbf{D}\right)=-\frac{1}{2}\left(\mathbf{D}-\mathbf{D}^T\right) \qquad QED$$

EXAMPLE 15 :

Decompose the matrix $\mathbf{A}=\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}$ into a symmetric and a skew-symmetric part.

• *Symmetric part*:

$$\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)=\frac{1}{2}\left(\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}+\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 0 \\ 3 & 5 & 1 \end{bmatrix}\right)=\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 2.5 \\ 3 & 2.5 & 1 \end{bmatrix}$$

• *Skew-symmetric part*:

$$\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^T\right)=\frac{1}{2}\left(\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}-\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 0 \\ 3 & 5 & 1 \end{bmatrix}\right)=\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2.5 \\ 0 & -2.5 & 0 \end{bmatrix}$$

♦

• **Definition :** Elementary Row Operations

Many discussions of matrices center around the *elementary row operations*. These represent simple operations which can be performed on the rows of a matrix $\mathbf{A}$ to yield a new matrix. The elementary row operations are:

1) multiply a row of $\mathbf{A}$ by a scalar m ;
2) interchange two rows in $\mathbf{A}$; and,
3) add m times one row to a second row of $\mathbf{A}$.

4. Partitioning of Matrices

In certain instances it is advantageous to partition a matrix into submatrices.

10. References

1. F. M. White, *Heat and Mass Transfer,* Addison Wesley, 1988.