MATERIAL HYPERBOLIC ELASTIC command

Synopsis

The MATERIAL HYPERBOLIC_ELASTIC command is used to specify parameters associated with various forms of the quasilinear (hyperbolic) model, originally proposed by Duncan and Chang in 1970 \cite{8}. The hyperbolic material idealization, though rather limited in scope \cite{6}, has none the less gained popularity in the field of geotechnical engineering. Consequently, various versions of this material idealization are available in APES.

Syntax

The following syntax is used to describe the MATERIAL HYPERBOLIC_ELASTIC idealization:

```
MATerial HYPerbolic_elastic NUMber #
   (DEScrition “string”)
   KINd DUNCAN_CHANG
   (MODULUS_NUMBER_Loading #.#) (MODULUS_NUMBER_Unloading #.#)
   (MODULUS_Exponent #.#)
   (FRIction_angle #.#) (COHesion #.#) (FAilure_ratio #.#)
   (ATMospheric_pressure #.#) (POISSons_ratio #.#)
   (RELaxation_parameter #.#)
```

```
MATerial HYPerbolic_elastic NUMber #
   (DEScrition “string”)
   KINd KULHAWY_DUNCAN
   (MODULUS_NUMBER_Loading #.#) (MODULUS_NUMBER_Unloading #.#)
   (MODULUS_Exponent #.#)
   (FRIction_angle #.#) (COHesion #.#) (FAilure_ratio #.#)
   (ATMospheric_pressure #.#)
   (POISSONS_NUMBER_G #.#) (POISSONS_NUMBER_F #.#) (POISSONS_NUMBER_d #.#)
   (RELaxation_parameter #.#)
```
MATerial HYPerbolic_elastic NUMber #
   (DEScription “string”)
   KINd HERRMANN
   (MODULUS_NUMBER_Loading #.#) (MODULUS_NUMBER_Unloading #.#)
      (MODULUS_Exponent #.#)
      (FRIction_angle #.#) (COHesion #.#) (FAilure_ratio #.#)
      (ATMospheric_pressure #.#) (BULK_modulus #.#)
      (RELaxation_parameter #.#)

MATerial HYPerbolic_elastic NUMber #
   (DEScription “string”)
   KINd DUNCAN_1980
   (MODULUS_NUMBER_Loading #.#) (MODULUS_NUMBER_Unloading #.#)
      (MODULUS_Exponent #.#)
      (FRIction_angle #.#) (COHesion #.#) (FAilure_ratio #.#)
      (ATMospheric_pressure #.#)
      (BULK_MODULUS_Number #.#) (BULK_MODULUS_Exponent #.#)
      (RELaxation_parameter #.#)
Explanatory Notes

A quasilinear elastic hyperbolic material idealization assumes that the soil does not experience stress or strain induced anisotropy; i.e., the soil may at all times be characterized by instantaneous values of the elastic (Young’s) modulus $E$ and Poisson’s ratio $\nu$. The keywords associated with the various versions of the quasilinear elastic hyperbolic material idealization are described below.

- The **NUMBER** keyword is used to specify the (global) number of the material associated with the incompressible isotropic elastic idealization. The *default* material number is one (1).
- The optional alphanumeric string associated with the **DESCRIPTION** keyword must be enclosed in double quotes ("."). It is used solely to describe the material being idealized to the analyst. The **DESCRIPTION** string is printed as part of the “echo” of the material information.

The **DUNCAN_CHANG** keyword is used to specify parameters associated with the original version of the hyperbolic model proposed by Duncan and Chang [8]. The following keywords are associated with this version of the model:

- The **MODULUS_NUMBER_LOADING** keyword is associated with the modulus number $K$ under conditions of loading. The *default* value is equal to 1.0.
- The **MODULUS_NUMBER_UNLOADING** keyword is associated with the modulus number $K_{ur}$ under conditions of unloading. The *default* value is equal to 1.0.
- **MODULUS_EXPONENT** refers to the exponent $n$ associated with the definition of the instantaneous value of the elastic modulus ($E_i$). The *default* value is equal to 1.0.
- The **FRICTION_ANGLE** keyword is used to specify the angle of internal friction; this angle must be given in *degrees*. The *default* value is 30.0 degrees.
- The **COHESION** keyword is used to specify the value of the cohesion intercept. The *default* intercept is 0.0.
- The **FAILURE_RATIO** keyword is used to specify the value of the failure ratio $R_f$; the *default* value is 0.8.
- The **ATMOSPHERIC_PRESSURE** keyword is used to specify the magnitude of the atmospheric pressure, which is used to normalize the confining stress $\sigma_3$. The *default* **ATMOSPHERIC_PRESSURE** is equal to 1.0.
- The **POISSONS_RATIO** keyword is used to specify the value of Poisson’s ratio ($\nu$). The *default** POISSONS_RATIO** is equal to 0.20.
- The **RELAXATION_PARAMETER** represents an attempt at accelerating the performance of the constitutive model [10]; the *default* value is 0.80.
The **KULHAWY_DUNCAN** keyword is used to specify parameters associated with the hyperbolic model proposed by Kulhawy and Duncan [18]. This version of the model attempted to improve upon the predictive capabilities of the original formulation, the latter being specified via the **DUNCAN_CHANG** keyword. The following keywords are associated with the **KULHAWY_DUNCAN** version of the model:

- **MODULUS_NUMBER_LOADING** keyword is associated with the modulus number $K$ under conditions of loading. The *default* value is equal to 1.0.

- **MODULUS_NUMBER_UNLOADING** keyword is associated with the modulus number $K_{ur}$ under conditions of unloading. The *default* value is equal to 1.0.

- **MODULUS_EXPONENT** refers to the exponent $n$ associated with the definition of the instantaneous value of the elastic modulus ($E_i$). The *default* value is equal to 1.0.

- The **FRICTION_ANGLE** keyword is used to specify the angle of internal friction; this angle must be given in *degrees*. The *default* value is 30.0 degrees.

- The **COHESION** keyword is used to specify the value of the cohesion intercept. The *default* intercept is 0.0.

- The **FAILURE_RATIO** keyword is used to specify the value of the failure ratio $R_f$; the *default* value is 0.8.

- The **ATMOSPHERIC_PRESSURE** keyword is used to specify the magnitude of the atmospheric pressure, which is used to normalize the confining stress $\sigma_3$. The *default* **ATMOSPHERIC_PRESSURE** is equal to 1.0.

- The **POISSONS_NUMBER_G** keyword is used to specify the value of the Kulhawy and Duncan [18] model parameter $G$. This parameter represents the value of the initial Poisson’s ratio at a pressure of one atmosphere. The *default** **POISSONS_NUMBER_G** is equal to 1.0.

- The **POISSONS_NUMBER_F** keyword is used to specify the value of the Kulhawy and Duncan [18] model parameter $F$. This parameter represents the rate of change of Poisson’s ratio with $\sigma_3$. The *default** **POISSONS_NUMBER_F** is equal to 1.0.

- The **POISSONS_NUMBER_D** keyword is used to specify the value of the Kulhawy and Duncan [18] model parameter $D$. This parameter expresses the rate of change of the tangential Poisson’s ratio ($\nu_t$) with strain. The *default** **POISSONS_NUMBER_D** is equal to 1.0.

- The **RELAXATION_PARAMETER** represents an attempt at accelerating the performance of the constitutive model [10]; the *default* value is 0.80.
The **HERRMANN** keyword is used to specify parameters associated with a slightly modified version of the hyperbolic model as proposed by Herrmann [10]. The following keywords are associated with this version of the model:

- The **MODULUS_NUMBER>Loading** keyword is associated with the modulus number $K$ under conditions of loading. The *default* value is equal to 1.0.

- The **MODULUS_NUMBER>Unloading** keyword is associated with the modulus number $K_{ur}$ under conditions of unloading. The *default* value is equal to 1.0.

- **MODULUS_EXPONENT** refers to the exponent $n$ associated with the definition of the instantaneous value of the elastic modulus ($E_i$). The *default* value is equal to 1.0.

- The **FRICTION_ANGLE** keyword is used to specify the angle of internal friction; this angle must be given in *degrees*. The *default* value is 30.0 degrees.

- The **COHESION** keyword is used to specify the value of the cohesion intercept. The *default* intercept is 0.0.

- The **FAILURE_RATIO** keyword is used to specify the value of the failure ratio $R_f$; the *default* value is 0.8.

- The **ATMOSPHERIC_PRESSURE** keyword is used to specify the magnitude of the atmospheric pressure, which is used to normalize the confining stress $\sigma_3$. The *default* **ATMOSPHERIC_PRESSURE** is equal to 1.0.

- The **BULK_MODULUS** keyword is used to specify the value of the elastic bulk modulus ($B$). The *default** BULK_MODULUS is equal to 1.0.

- The **RELAXATION_PARAMETER** represents an attempt at accelerating the performance of the constitutive model [10]; the *default* value is 0.80.
The keyword **DUNCAN.1980** signifies another modified version of the hyperbolic model as proposed by Duncan [7]. The following keywords are associated with this version of the model:

- The **MODULUS_NUMBER_LOADING** keyword is associated with the modulus number $K$ under conditions of loading. The *default* value is equal to 1.0.
- The **MODULUS_NUMBER_UNLOADING** keyword is associated with the modulus number $K_{ur}$ under conditions of unloading. The *default* value is equal to 1.0.
- **MODULUS_EXPONENT** refers to the exponent $n$ associated with the definition of the instantaneous value of the elastic modulus ($E_i$). The *default* value is equal to 1.0.
- The **FRICTION_ANGLE** keyword is used to specify the angle of internal friction; this angle must be given in degrees. The *default* value is 30.0 degrees.
- The **COHESION** keyword is used to specify the value of the cohesion intercept. The *default* intercept is 0.0.
- The **FAILURE_RATIO** keyword is used to specify the value of the failure ratio $R_f$; the *default* value is 0.8.
- The **ATMOSPHERIC_PRESSURE** keyword is used to specify the magnitude of the atmospheric pressure, which is used to normalize the confining stress $\sigma_3$. The *default* **ATMOSPHERIC_PRESSURE** is equal to 1.0.
- The **BULK_MODULUS_NUMBER** keyword is used to specify the value of the “bulk modulus number” ($K_b$) introduced by Duncan [7]. The *default** **BULK_MODULUS_NUMBER** is equal to 1.0.
- The **BULK_MODULUS_EXPONENT** keyword is used to specify the value of the “bulk modulus exponent” ($n$) introduced by Duncan [7]. The *default** **BULK_MODULUS_EXPONENT** is equal to 1.0.
- The **RELAXATION_PARAMETER** represents an attempt at accelerating the performance of the constitutive model [10]; the *default* value is 0.80.
Example of Command Usage

Duncan-Chang Material Idealization

A Duncan and Chang [8] quasilinear elastic hyperbolic material idealization is to be used with the following model parameter values: \( K = 200 \), \( K_{ur} = 350 \), \( n = 0.43 \), \( R_f = 0.85 \), \( \phi = 17^\circ \), \( c = 33.0 \) kPa and \( \nu = 0.40 \). To describe such a material idealization for the APES computer program, enter the following command:

```
material hyperbolic_elastic number 2 &
description "idealization of sand fill" kind "duncan_chang" &
modulus_number_load 200.0 modulus_number_unload 350.0 &
modulus_exponent 0.43 failure_ratio 0.85 friction 17.0 &
cohesion 33.0 poissons_ratio 0.40
```

Kulhawy-Duncan Material Idealization

A Kulhawy and Duncan [18] quasilinear elastic hyperbolic material idealization is to be used with: \( K = 1580 \), \( K_{ur} = 1900 \), \( n = 1.05 \), \( R_f = 0.90 \), \( \phi = 42.7^\circ \), \( c = 0.0 \) kPa and \( G = 0.46 \), \( F = 0.17 \) and \( D = 24.0 \). To describe such a material idealization, enter the following command:

```
material hyperbolic_elastic duncan_chang number 2 &
description "idealization of sand fill" &
kind "kulhawy_duncan" &
modulus_number_load 1580.0 modulus_number_unload 1900.0 &
modulus_exponent 1.05 failure_ratio 0.90 friction 42.7 &
poissons_number_G 0.46 poissons_number_F 0.17 &
poissons_number_D 24.0
```

Duncan-Chang Material Idealization of Dense Sacramento River Sand

The following commands are used to specify the material parameters associated with the Duncan-Chang [8] version of the quasilinear elastic (hyperbolic) material idealization as applied to dense Sacramento River sand:

```
material hyperbolic_elastic num 1 &
desc "parameters for DENSE Sacramento River sand" &
kind DUNCAN_CHANG & modulus_number_load 1600.0 &
modulus_number_unload 1600.0 & modulus_exp 0.45 &
friction_angle 41.0 cohesion 0.0 &
failure_ratio 0.95 &
atmospheric_pre 101.4 &
poissons_ratio 0.20
```
Kulhawy-Duncan Material Idealization of Dense Sacramento River Sand

The following commands are used to specify the material parameters associated with the Kulhawy-Duncan [18] version of the quasilinear elastic (hyperbolic) material idealization as applied to dense Sacramento River sand:

```plaintext
material hyperbolic_elastic num 2 &
desc "parameters for DENSE Sacramento River sand" &
kind Kulhawy_DUNCAN &
modulus_number_load 1600.0 &
modulus_number_unload 1600.0 &
modulus_exp 0.45 &
friction_angle 41.0 cohesion 0.0 &
failure_ratio 0.95 &
atmospheric_pre 101.4 &
poissons_number_G 0.450 poissons_number_F -0.10 poissons_number_D 0.20
```

Duncan Material Idealization of Dense Sacramento River Sand

The following commands are used to specify the material parameters associated with the Duncan [7] version of the quasilinear elastic (hyperbolic) material idealization as applied to dense Sacramento River sand:

```plaintext
material hyperbolic_elastic num 3 desc "parameters for DENSE Sacramento River sand" &
kind DUNCAN_1980 &
modulus_number_load 1600.0 & modulus_number_unload 1600.0 &
modulus_exp 0.45 &
friction_angle 41.0 &
failure_ratio 0.95 &
bulk_modulus_number 500.0 &
bulk_modulus_exp 1.58 & atmospheric_pre 101.4
```
Model Parameter Values

Some values of model parameter appearing in the literature are listed below. Further parameter sets are available in reports by Wong and Duncan [19] and Duncan et al. [9], as well as in sundry papers [3].

Basic Model of Duncan and Chang

Some values of the parameters associated with the original form of the quasilinear elastic “hyperbolic” model of Duncan and Chang [8] are listed below.

♣ Soil: Ottawa Sand

Description

- Subrounded Ottawa silica sand
- $G_s = 2.65$
- Dry density (air dried): $\gamma_d = 107 \text{ lb/ft}^3$

Parameter Values

- $K = 1200.0$
- $K_{ur} = \text{n/a}$
- $n = 0.66$
- $R_f = 0.88$
- $\phi = 38.0^\circ$
- $c = 0.0$
- $\nu = 0.0$

Reference

♣ Soil: Ottawa Sand

Description
• Subrounded Ottawa silica sand
• $G_s = 2.65$
• Dry density (air dried): $\gamma_d = 107 \, \text{lb/ft}^3$

Parameter Values

• $K = 1355.2$
• $K_{ur} = \text{n/a}$
• $n = 0.61$
• $R_f = 0.91$
• $\phi = 38.4^\circ$
• $c = 0.0$
• $\nu = 0.30$

Reference

♣ Soil: Sand

Description

• $\gamma_d = 17.7 \, \text{kN/m}^3$

Parameter Values

• $K = 1000.0$
• $K_{ur} = \text{n/a}$
• $n = 1.00$
• $R_f = 0.90$
• $\phi = 38.0^\circ$
• $c = 0.0$
• $\nu = 0.33$

Reference

♣ Soil: Sand Fill

Parameter Values
• $K = 580.0$
• $K_{ur} = n/a$
• $n = 0.50$
• $R_f = 0.85$
• $\phi = 36.0^\circ$
• $c = 0.0$
• $\nu = 0.30$

Reference

♣ Soil: Sand Fill

Parameter Values
• $K = 177.0$
• $K_{ur} = 354.0$
• $n = 0.43$
• $R_f = 0.85$
• $\phi = 17.0^\circ$
• $c = 33.0 \, kPa$
• $\nu = 0.40$
Reference

♣ Soil: Sandy Clay

Description

- Dark grey, slightly silty, fine to coarse sand and clay mixture.
- $G_s = 2.66$
- Liquid limit = 37%, Plastic limit = 19%
- Water content = 19.3%
- Dry density (air dried): $\gamma_d = 100.7 \text{ lb/ft}^3$

Parameter Values

- $K = 290.7$
- $K_{ur} = \text{n/a}$
- $n = 0.41$
- $R_f = 0.96$
- $\phi = 14.2^\circ$
- $c = 7.45 \text{ psi}$
- $\nu = 0.30$.

Reference

♣ Soil: Sandy Clay

Description

- Density: $\gamma = 124.6 \text{ lb/ft}^3$
- Liquid limit = 25%, Plastic limit = 12%


- \( D_{30} = 0.002 \ mm \), \( D_{60} = 0.040 \ mm \)

### Parameter Values

- \( K = 280.0 \)
- \( K_{ur} = 840 \)
- \( n = 0.6 \)
- \( R_f = 0.93 \)
- \( \phi = 18.0^\circ \)
- \( c = 12.64 \ psi \)
- \( \nu = 0.3 \) (assumed)

### Reference


- ♦ Soil: Sandy Silt

### Description

- Density: \( \gamma = 121.3 \ lb/ft^3 \)
- Liquid limit = 19%, Plastic limit = 1%
- \( D_{10} = 0.013 \ mm \)
- \( D_{30} = 0.045 \ mm \)
- \( D_{60} = 0.070 \ mm \)

### Parameter Values

- \( K = 100.0 \)
- \( K_{ur} = 300 \)
- \( n = 0.84 \)
- \( R_f = 0.77 \)
- \( \phi = 27.0^\circ \)
• $c = 7.5 \text{ psi}$

• $\nu = 0.3$ (assumed)

**Reference**

♣ **Soil: Geosynthetic-Reinforced, Pile-Supported Earth Platforms**

**Parameter Values**

For foundation soil:

• $K = 50$

• $n = 0.45$

• $R_f = 0.7$

• $c = 0 \text{ kPa}$

• $\phi = 22^\circ$

• $\nu = 0.30$

For embankment fill:

• $K = 150$

• $n = 0.25$

• $R_f = 0.7$

• $c = 0 \text{ kPa}$

• $\phi = 30^\circ$

• $\nu = 0.30$

**Reference**

♣ **Soil: Geosynthetic-Encased Stone Columns**

**Description**
• Density of stone column material: $\gamma = 20 \text{kN/m}^3$
• Density of foundation soil: $\gamma = 17 \text{kN/m}^3$
• Density of embankment fill: $\gamma = 17 \text{kN/m}^3$

**Parameter Values**

For stone column:

• $K = 1200$
• $n = 0.7$
• $R_f = 0.7$
• $c = 0 \text{kPa}$
• $\phi = 42^\circ$
• $\nu = 0.30$

For foundation soil:

• $K = 50$
• $n = 0.5$
• $R_f = 0.7$
• $c = 20 \text{kPa}$
• $\phi = 0^\circ$
• $\nu = 0.45$

For embankment fill:

• $K = 150$
• $n = 0.5$
• $R_f = 0.7$
• $c = 0 \text{kPa}$
• $\phi = 30^\circ$
• $\nu = 0.30$

**Reference**

Model of Kulhawy and Duncan

Some values of the parameters associated with the quasilinear elastic hyperbolic model proposed by Kulhawy and Duncan [18] are listed below.

♣ Soil: Estuarine Cohesive Soil

Description

• Dry density: $\gamma_d = 94 \text{ lb/ft}^3$
• Liquid limit = 29%, Plastic limit = 9%.
• Parameter values were determined from the results of drained triaxial tests.

Parameter Values

• $K = 150.0$
• $K_{ur} = \text{n/a}$
• $n = 0.65$
• $R_f = 0.87$
• $\phi = 34.0^\circ$
• $c = 0.0 \text{ psf}$
• $G = 0.21$
• $F = -0.02$
• $d = 6.8$

Reference

♣ Soil: Estuarine Cohesionless Soil

Description

• Dry density: $\gamma_d = 94 \text{ lb/ft}^3$
• Parameter values were determined from the results of drained triaxial tests.
Parameter Values

- $K = 260.0$
- $K_{ur} = n/a$
- $n = 0.69$
- $R_f = 0.69$
- $\phi = 38.0^o$
- $c = 0.0 \text{ psf}$
- $G = 0.24$
- $F = -0.02$
- $d = 13.4$

Reference

♣ Soil: Older Marine Cohesive Soil

Description

- Dry density: $\gamma_d = 106 \text{ lb/ft}^3$
- Liquid limit = 27%, Plastic limit = 10%.
- Parameter values were determined from the results of drained triaxial tests.

Parameter Values

- $K = 150.0$
- $K_{ur} = n/a$
- $n = 0.65$
- $R_f = 0.84$
- $\phi = 34.0^o$
- $c = 0.0 \text{ psf}$
- $G = 0.42$
• $F = 0.42$
• $d = 3.2$

Reference

♣ Soil: Older Marine Cohesionless Soil

Description
• Dry density: $\gamma_d = 106 \text{ lb/ft}^3$
• Parameter values were determined from the results of drained triaxial tests.

Parameter Values
• $K = 190.0$
• $K_{ur} = \text{n/a}$
• $n = 0.90$
• $R_f = 0.67$
• $\phi = 37.0^\circ$
• $c = 0.0 \text{ psf}$
• $G = 0.32$
• $F = 0.18$
• $d = 8.7$

Reference

♣ Soil: Estuarine Cohesive Soil

Description
• Dry density: $\gamma_d = 94 \text{ lb/ft}^3$
• Liquid limit = 31%
• Plastic limit = 10%
• Parameter values were determined from the results of undrained triaxial tests.

**Parameter Values**

- $K = 165.0$
- $K_{ur} = n/a$
- $n = 1.10$
- $R_f = 0.93$
- $\phi = 19.0^\circ$
- $c = 460.0 \text{ psf}$

**Reference**


♣ **Soil: Older Marine Cohesive Soil**

**Description**

- Dry density: $\gamma_d = 106 \text{ lb/ft}^3$
- Liquid limit = 31%, Plastic limit = 10%
- Parameter values were determined from the results of undrained triaxial tests.

**Parameter Values**

- $K = 315.0$
- $K_{ur} = n/a$
- $n = 0.43$
- $R_f = 0.90$
- $\phi = 19.0^\circ$
- $c = 460.0 \text{ psf}$
Reference

♣ Soil: Sandy Gravel (stone column material)

Description

- Dry density: $\gamma_d = 116 \text{ lb/ft}^3$
- Parameter values were determined from the results of drained triaxial tests.

Parameter Values

- $K = 390.0$
- $K_{ur} = \text{n/a}$
- $n = 0.59$
- $R_f = 0.86$
- $\phi = 41.0^\circ$
- $c = 0.0 \text{ psf}$
- $G = 0.34$
- $F = 0.20$
- $d = 12.0$

Reference

♣ Soil: Leighton Buzzard Sand

Parameter Values

- $K = 1573.0$
- $K_{ur} = 1890.0$
- $n = 1.05$
- $R_f = 0.90$
• $\phi = 42.7^\circ$
• $c = 0.0$
• $G = 0.46$
• $F = 0.17$
• $D = 24.0$

Reference

♣ Soil: Ottawa Sand

Description

• Subrounded Ottawa silica sand
• $G_s = 2.65$
• Dry density (air dried): $\gamma_d = 107 \text{ lb/ft}^3$

Parameter Values

• $K = 908.0$
• $K_{ur} = \text{n/a}$
• $n = 0.59$
• $R_f = 0.86$
• $\phi = 38.0^\circ$
• $c = 0.0$
• $G = 0.33$
• $F = 0.01$
• $d = 12.0$
Reference

♣ Soil: Sandy Clay

Description
- Dark grey, slightly silty, fine to coarse sand and clay mixture.
- $G_s = 2.66$
- Liquid limit = 37%
- Plastic limit = 19%
- Water content = 19.3%
- Dry density (air dried): $\gamma_d = 100.7 \text{ lb/ft}^3$

Parameter Values
- $K = 144.0$
- $K_{ur} = 100.0$
- $n = 0.06$
- $R_f = 0.87$
- $\phi = 26.0^\circ$
- $c = 2.0 \text{ psi}$
- $G = 0.20$
- $F = 0.10$
- $d = 4.80$

Reference
• Dark grey, slightly silty, fine to coarse sand and clay mixture.
• $G_s = 2.66$
• Liquid limit = 37%, Plastic limit = 19%
• Water content = 19.3%
• Dry density (air dried): $\gamma_d = 100.7 \text{ lb/ft}^3$

Parameter Values

• $K = 144.0$
• $K_{ur} = \text{n/a}$
• $n = 0.06$
• $R_f = 0.83$
• $\phi = 26.0^\circ$
• $c = 2.0 \text{ psi}$
• $G = 0.40$
• $F = 0.10$
• $d = 4.80$

Reference
Model of Duncan

Some values of the parameters associated with the quasilinear elastic hyperbolic model proposed by Duncan [7] are listed below.

♣ Soil: Clean Ottawa sand

Parameter Values

- $K = 600.0$
- $K_{ur} = 600.0$
- $n = 0.40$
- $R_f = 0.80$
- $\phi = 35.0^\circ$
- $c = 1.0 \, kPa$
- $K_b = 300.0$
- $m = 0.40$

Reference

♣ Soil: Leighton Buzzard sand

Description

- Density: $\gamma = 17.0 \, kN/m^3$

Parameter Values

- $K = 950.0$
- $K_{ur} = n/a$
- $n = 0.80$
- $R_f = 0.70$
- $\phi = 49.6^\circ$
• $c = 0.0 \ kPa$

• $K_b = 250.0$

• $m = 0.460$

Reference

♣ Soil: Ottawa sand

Parameter Values

• $K = 400.0$

• $K_{ur} = 2000.0$

• $n = 0.85$

• $R_f = 0.86$

• $\phi = 43.0^\circ$

• $c = 0.5 \ psi$

• $K_b = 700.0$

• $m = 0.50$

Reference

♣ Soil: Ottawa Sand

Description

• Subrounded Ottawa silica sand

• $G_s = 2.65$

• Dry density (air dried): $\gamma_d = 107 \ lb/ft^3$

Parameter Values
• $K = 1116.0$

• $K_{ur} = 1500.0$

• $n = 0.66$

• $R_f = 0.87$

• $\phi = 38.4^\circ$

• $c = 0.50 \text{ psi}$

• $K_b = 907.0$

• $m = 0.0$

Reference


♣ Soil: Ottawa Sand

Description

• Subrounded Ottawa silica sand

• $G_s = 2.65$.

• Dry density (air dried): $\gamma_d = 107 \text{ lb/ft}^3$

Parameter Values

• $K = 1000.0$

• $K_{ur} = 1500.0$

• $n = 0.67$

• $R_f = 0.85$

• $\phi = 38.0^\circ$

• $c = 0.0 \text{ psi}$

• $K_b = 700.0$

• $m = 0.40$
Reference

♣ Soil: Ottawa Sand

Description
- Subrounded Ottawa silica sand
- $G_s = 2.65$
- Dry density (air dried): $\gamma_d = 107 \text{ lb/ft}^3$

Parameter Values
- $K = 950.0$
- $K_{ur} = n/a$
- $n = 0.36$
- $R_f = 0.75$
- $\phi = 38.0^\circ$
- $c = 0.0 \text{ psi}$
- $K_b = 485.0$
- $m = 0.64$

Reference

♣ Soil: RMC Sand

Parameter Values
- $K = 950.0$
- $K_{ur} = n/a$
- $n = 0.50$
• $R_f = 0.75$
• $\phi = 53.0^\circ$
• $c = 0.0 \text{ psi}$
• $K_b = 250.0$
• $m = 0.65$

Reference

♣ Soil: Sand

Parameter Values
• $K = 913.0$
• $K_{ur} = 1485.0$
• $n = 0.60$
• $R_f = 0.64$
• $\phi = 46.1^\circ$
• $c = 0.0 \text{ psi}$
• $K_b = 250.0$
• $m = 0.80$

Reference

♣ Soil: Sandy Clay

Description
• Dark grey, slightly silty, fine to coarse sand and clay mixture.
• $G_s = 2.66$. 
- Liquid limit = 37%, Plastic limit = 19%.
- Water content = 19.3%.
- Dry density (air dried): $\gamma_d = 100.7 \text{ lb/ft}^3$.

**Parameter Values**

- $K = 370$, $K_{ur} = 500$, $n = 0.28$, $R_f = 0.98$, $\phi = 15.0^\circ$, $c = 7.47 \text{ psi}$, $K_b = 210.0$, $m = 0.0$.

**Reference**


♣ **Soil: Sandy Clay**

**Description**

- Dark grey, slightly silty, fine to coarse sand and clay mixture.
- $G_s = 2.66$.
- Liquid limit = 37%, Plastic limit = 19%.
- Water content = 19.3%.
- Dry density (air dried): $\gamma_d = 100.7 \text{ lb/ft}^3$.

**Parameter Values**

- $K = 200.0$, $K_{ur} = n/a$, $n = 0.50$, $R_f = 0.92$, $\phi = 26.3^\circ$, $c = 2.0 \text{ psi}$, $K_b = 600.0$, $m = 0.48$.

**Reference**


♣ **Soil: Three Clays**

For Clay “X”:

- $K = 55.0$, $n = 0.0$, $R_f = 0.95$, $c = 0.0 \text{ psi}$, $\phi = 30^\circ$, $K_b = 45.0$, $m = 0.0$.

For Clay “Y”:
• $K = 165.0$, $n = 0.0$, $R_f = 0.60$, $c = 8.0$ psi, $\phi = 0^\circ$, $K_b = 80.0$, $m = 0.0$.

For Kaolinite:

• $K = 800.0$, $n = 0.0$, $R_f = 0.72$, $c = 28.5$ psi, $\phi = 0^\circ$, $K_b = 10000.0$, $m = 0.0$.

Reference
Theoretical Considerations: Quasilinear Elastic Models Based on a Hyperbolic Relation

Konder and his co-workers \cite{12, 13, 14, 15} approximated stress-strain curves for both clays and sands by the form shown in Figure 1. Analytically, such curves are represented by the following *hyperbolic* relation:

\[
\sigma_1 - \sigma_3 = \frac{\varepsilon_1}{a + b\varepsilon_1}
\]  \hspace{1cm} (1)

where \( \sigma_1 \) and \( \sigma_3 \) are the major and minor principal stresses, respectively, \( \varepsilon_1 \) is the major principal strain, and \( a \) and \( b \) are model parameters whose values are determined by fitting experimental data.

Assuming \( \sigma_3 \) constant, the tangent modulus \( E_t \) is given by

\[
E_t = \frac{\partial (\sigma_1 - \sigma_3)}{\partial \varepsilon_1} = \frac{(a + b\varepsilon_1)(1) - b\varepsilon_1}{(a + b\varepsilon_1)^2} = \frac{a}{(a + b\varepsilon_1)^2} \]  \hspace{1cm} (2)

Solving equation (1) for \( \varepsilon_1 \) gives

\[
\varepsilon_1 = \frac{a(\sigma_1 - \sigma_3)}{1 - b(\sigma_1 - \sigma_3)} \]  \hspace{1cm} (3)

Substituting this expression into equation (2) gives

\[
E_t = \frac{1}{a} \left[1 - b(\sigma_1 - \sigma_3)\right]^2 \]  \hspace{1cm} (4)
We next note that the initial tangent modulus $E_i$ is the slope of the stress-strain curve when $(\sigma_1 - \sigma_3) = 0$. Evaluating equation (4) for this condition gives

$$E_i = \frac{1}{a} [1 - b(0)]^2 \Rightarrow a = \frac{1}{E_i} \quad (5)$$

In addition, we note that as $(\sigma_1 - \sigma_3) \to (\sigma_1 - \sigma_3)_{ult}$, $E_t \to 0$. Here $(\sigma_1 - \sigma_3)_{ult}$ is the asymptotic value of the principal stress difference at “infinite” strain (Figure 1). Evaluating equation (4) for this condition gives

$$E_t = \frac{1}{a} [1 - b(\sigma_1 - \sigma_3)_{ult}]^2 = 0 \Rightarrow b = \frac{1}{(\sigma_1 - \sigma_3)_{ult}} \quad (6)$$

Substituting the expressions for $a$ and $b$ into equation (4) gives the following expression for $E_t$:

$$E_t = E_i \left[1 - \frac{(\sigma_1 - \sigma_3)}{(\sigma_1 - \sigma_3)_{ult}}\right]^2 \quad (7)$$

Clearly, as $(\sigma_1 - \sigma_3) \to (\sigma_1 - \sigma_3)_{ult}$, $E_t \to 0$.

Remarks

1. The fact that equation (1) involves only $\varepsilon_1$ and the principal stress difference implies an axisymmetric triaxial material state (i.e., $\sigma_2 = \sigma_3$, $\varepsilon_2 = \varepsilon_3$).

2. From the form of the stress-strain curve shown in Figure 1, it is evident that the hyperbolic idealization can only be accurate for normally consolidated or lightly overconsolidated clays and for loose sands; that is, in cases where there is no appreciable drop from a peak principal stress difference to a residual value.

3. The hyperbolic idealization says nothing about the volume change of the material. Indeed, since the model assumes elastic isotropy, it follows that no volume changes will be produced upon shearing, which is counter to experimental observations.

4. If equation (1) is re-written as

$$\frac{\varepsilon_1}{\sigma_1 - \sigma_3} = b\varepsilon_1 + a \quad (8)$$

and if we view $\varepsilon_1$ as the abscissa and $\varepsilon_1/(\sigma_1 - \sigma_3)$ as the ordinate, then the hyperbolic stress-strain idealization plots as a straight line (Figure 2). The slope of this line is $b$, and the intercept is $a$.

Model of Duncan and Chang

Citing the previous experimental findings of Janbu [11], Duncan and Chang [8] noted that each of the constants $a$ and $b$ should be dependent on the minor principal (confining) effective stress $\sigma_3$. More precisely, they suggested that $E_i$ vary in the following manner:

$$E_i = \frac{1}{a} = K p_a \left(\frac{\sigma_3}{p_a}\right)^n \quad (9)$$

where $K$ is a dimensionless “modulus number,” $p_a$ is the atmospheric pressure (with the same units as $\sigma_3$), and $n$ is a dimensionless “modulus exponent” that determines the rate of variation of $E_i$ with $\sigma_3$. 
Duncan and Chang further suggested the following relation between principal stress difference at failure (or the “compressive strength”) \((\sigma_1 - \sigma_3)_f\) and its ultimate value:

\[
(\sigma_1 - \sigma_3)_f = R_f (\sigma_1 - \sigma_3)_{ult} \tag{10}
\]

where \(R_f\) is a “failure ratio” \((R_f \leq 1.0)\).

The principal stress difference at failure was assumed to be related to the minor principal (confining) stress through the Mohr-Coulomb failure criterion; viz.,

\[
(\sigma_1 - \sigma_3)_f = \frac{2(c \cos \phi + \sigma_3 \sin \phi)}{1 - \sin \phi} \tag{11}
\]

where \(c\) is the cohesion intercept and \(\phi\) is the angle of internal friction.

Using equations (10) and (11), it follows that

\[
b = \frac{1}{(\sigma_1 - \sigma_3)_{ult}} = \frac{R_f}{(\sigma_1 - \sigma_3)_f} = \frac{R_f(1 - \sin \phi)}{2(c \cos \phi + \sigma_3 \sin \phi)} \tag{12}
\]

Substituting equations (9) and (12) into the first of equation (7) gives

\[
E_t = Kp_a \left(\frac{\sigma_3}{p_a}\right)^n \left[1 - \frac{R_f(1 - \sin \phi)(\sigma_1 - \sigma_3)}{2(c \cos \phi + \sigma_3 \sin \phi)}\right]^2 \tag{13}
\]

In general, the value of \(E_t\) will be too low to properly account for unloading. Consequently, based on the results of a number of loading-unloading-reloading tests on sands, Duncan and Chang [8] proposed the following relationship for the elastic modulus associated with unloading and reloading:

\[
E_{ur} = K_{ur}p_a \left(\frac{\sigma_3}{p_a}\right)^n \tag{14}
\]
where $E_{ur}$ is the unloading-reloading modulus, and $K_{ur}$ is the corresponding dimensionless modulus number. Thus, for histories involving loading, unloading and reloading, the same value ($E_{ur}$) of the elastic modulus is used. Admittedly this is a simplification of the actual material behavior.

Experimental results indicate that, in general, $E_{ur} > E_t$. In particular, based on the results of a large number of laboratory experiments, Wong and Duncan [19] concluded that $K_{ur}$ was 1 to 3 times higher than $K_i$, whereas $n$ was essentially identical for initial loading and for unloading-reloading.

Duncan and Chang [8] assumed the second elastic constant, Poisson’s ratio ($\nu$), to be constant. Kulhawy and Duncan [18] subsequently modified this assumption in the manner described in the next section.

**Modifications Proposed by Kulhawy and Duncan**

While maintaining the same expressions for $E_i$, $E_t$ and $E_{ur}$ as presented by Duncan and Chang [8], Kulhawy and Duncan [18] proposed the following relation between the axial strain $\varepsilon_a$ and the radial strain $\varepsilon_r$:

$$\varepsilon_a = -\frac{\varepsilon_r}{f + D\varepsilon_r} \quad (15)$$

where $f$ and $D$ are model parameters. Noting that the tangent Poisson’s ratio is defined by

$$\nu_t = -\frac{\partial \varepsilon_r}{\partial \varepsilon_a} = \frac{f}{(1 - D\varepsilon_a)^2} \quad (16)$$

we see that $f$ equals the initial value $\nu_i$ of the tangent Poisson’s ratio at zero strain and $d$ expresses the rate of change of $\nu_i$ with strain.

Noting that $\nu_i$ generally decreases with increasing confining stress $\sigma_3$, Kulhawy and Duncan proposed the following expression:

$$\nu_i = G - F \log \left( \frac{\sigma_3}{p_a} \right) \equiv f \quad (17)$$

where $G$ is the value of $\nu_i$ at $\sigma_3 = p_a$ and $F$ is the decrease of $\nu_i$ for a ten-fold increase in $\sigma_3$. Substituting equation (17) into (16) gives

$$\nu_t = \frac{G - F \log \left( \frac{\sigma_3}{p_a} \right)}{(1 - D\varepsilon_a)^2} \quad (18)$$

where $D$ is a model parameter.

From equation (1) it follows that the major principal strain is

$$\varepsilon_1 = \frac{a(\sigma_1 - \sigma_3)}{1 - b(\sigma_1 - \sigma_3)} \quad (19)$$

Assuming that $\varepsilon_1 = \varepsilon_a$, and using equations (9) and (12) leads to

$$\varepsilon_1 = \frac{(\sigma_1 - \sigma_3)}{K p_a \left( \frac{\sigma_3}{p_a} \right)^n \left[ 1 - \frac{R_f(1 - \sin \phi)(\sigma_1 - \sigma_3)}{2(c \cos \phi + \sigma_3 \sin \phi)} \right]} \quad (20)$$
Equation (18), with \( \varepsilon_1 = \varepsilon_a \) given by equation (20), is thus used to compute the tangent Poisson’s ratio; viz.,

\[
\nu_t = \frac{G - F \log \left( \frac{\sigma_3}{P_a} \right)}{1 - \frac{D (\sigma_1 - \sigma_3)}{K P_a \left( \frac{\sigma_3}{P_a} \right)^n \left[ 1 - \frac{R_f (1 - \sin \phi) (\sigma_1 - \sigma_3)}{2(c \cos \phi + \sigma_3 \sin \phi)} \right]^2}}.
\] (21)

The description of Poisson’s ratio is thus controlled by values of the three model parameters \( G \), \( F \) and \( D \).

**Modification Proposed by Herrmann**

Herrmann [10] presented a slightly modified version of the Duncan-Chang model. The modification was made in order to avoid a serious problem that was observed in a number of analyses. In particular, the problem occurred for regions where the confining pressure \( \sigma_3 \) was sufficiently low to yield a low modulus, but not low enough to yield a high Poisson’s ratio. The result was that the soil in these regions experienced large decreases in volume; i.e., it effectively disappeared, thus violating the conservation of mass. To prevent this problem, the soil was assumed to possesses a constant bulk modulus \( B \). The Poisson’s ratio was then calculated from the following expression:

\[
\nu_t = \frac{1}{2} \left( 1 - \frac{E_t}{3B} \right).
\] (22)

A check is commonly made to ensure that \( 0 \leq \nu_t \leq 0.49 \).

**Modification Proposed by Duncan**

While maintaining the same expressions for \( E_i \), \( E_t \) and \( E_{ur} \) as presented by Duncan and Chang [8], Duncan [7] assumed that \( B \) depends only on the confining stress \( \sigma_3 \) and is independent of the principal stress difference. The following equation for the tangent elastic bulk modulus was thus proposed:

\[
B_t = K_b p_a \left( \frac{\sigma_3}{P_{atm}} \right)^m.
\] (23)

where \( K_b \) is a dimensionless “bulk modulus number” (equal to the value of \( B/p_a \) at \( \sigma_3 = p_a \)) and \( m \) is a dimensionless “bulk modulus exponent” (equal to the change in \( B/p_a \) for a ten-fold increase in \( \sigma \)). For the case of loading-unloading-reloading histories, no modification of equation (23) was proposed by Duncan [7].

Using equation (23) for the bulk modulus, Poisson’s ratio is computed from

\[
\nu_t = \frac{1}{2} \left( 1 - \frac{E_t}{3B_t} \right)
\] (24)

A check is commonly made to ensure that \( 0 \leq \nu_t \leq 0.49 \).
Advantages and Limitations of Hyperbolic-Based Relations

The advantages of quasilinear elastic models based on the hyperbolic stress-strain relationship stem from their simplicity and their successful use in analyzing a number of different practical problems. More precisely, the advantages of such models are:

1. The values of the required parameters can be readily determined from the results of a series of conventional axisymmetric triaxial compression tests. Use of this model thus does not require more extensive or more exotic testing programs than those which are routinely performed for other soil engineering purposes. This is an important consideration, because the cost of a few unusual or unconventional laboratory tests may easily exceed the cost of the finite element analysis for which they are performed.

2. The parameters associated with the model have readily understood physical significance. This is helpful because engineering judgment is enhanced when each element of the model is comprehensible in physical terms, and the effects of changes in the parameter values can be readily anticipated [7].

3. Parameter values have been determined for many soils, under both drained and undrained conditions. These data are useful for estimating parameter values when only soil type and density are known, and for judging the correctness of laboratory test results by means of comparisons with results for similar soils. Some parameter values reported in the literature are given in the following section.

4. The capabilities and limitations of the formulation are rather thoroughly documented and well-understood.

5. Quasilinear models are relatively easy to implement into computer programs. This implementation typically involves an incremental/iterative solution scheme.

Since models based on a hyperbolic relation represent a highly idealized characterization of the soil, they possess some rather significant limitations. Since these limitations should be understood by anyone using the model, they are summarized below.

1. In the model no account is made for the value of the intermediate principal stress \( \sigma_2 \). Thus the accuracy of predictions under other than axisymmetric triaxial conditions (e.g., plane strain, true triaxial, etc.) may be questionable.

2. Since a Mohr-Coulomb failure criterion is assumed, the failure envelope will be straight. For many soils this is known not to be the case, however. It is pertinent to note that Andrawes et al. [1] attempted to include the effect of the curvature of the Mohr failure envelope for cohesionless soils. They proposed the following relation:

\[
\phi = \phi_0 + \Delta \phi \log \left( \frac{\sigma_3}{P_a} \right)
\]  

(25)

where \( \phi_0 \) is the value of the effective friction angle for \( \sigma_3 \) equal to \( P_a \), and \( \Delta \phi \) is a material constant equal to the reduction in \( \phi \) for a ten-fold increase in \( \sigma_3 \).
3. Being based on the generalized Hooke’s Law the relationships are most suitable for analysis of stresses and movements prior to failure. The relationships are capable of predicting accurately non-linear relationships between loads and movements, and it is possible to continue the analyses up to the stage where some elements experience local failure. However, when a stage is reached where the behavior of the soil mass is controlled to a large extent by the properties assigned to elements which have already failed, the results will no longer be reliable, and they may be unrealistic in terms of the behavior of real soils at and after failure. These relationships are not useful, therefore, for analyses extending up to the stage of instability of a soil mass. They are useful for predicting movements in stable earth masses.

4. Special considerations must typically be included in order to properly model unloading.

5. The hyperbolic relationships do not include volume change due to changes in shear stress, or “shear dilatancy.” They may, therefore, be limited in the accuracy with which they can be used to predict deformations in dilatant soils such as dense sands under low confining pressures. To address this shortcoming, Bathurst and Karpurapu [2] extended the model of Duncan [7] by incorporating a dilation angle to simulate the dilation of soils.

6. For some combinations of parameter values and stresses, the values of tangent Poisson’s ratio \( \nu_t \) calculated may exceed 0.5. However, since thermodynamic considerations limit Poisson’s ratio to the range \(-1.0 \leq \nu \leq 0.50\), in most computer programs that employ these parameters, \( \nu_t \) is set equal to 0.49 under such circumstances.

7. The parameters are not fundamental soil properties, but only values of empirical coefficients which represent the behavior of the soil under a limited range of conditions. The values of the parameters depend on the density of the soil, its water content, the range of pressures used in testing, and the drainage conditions. In order that the parameters will be representative of the behavior of the soil under field conditions, the laboratory test conditions must correspond to the field conditions with regard to these factors.

8. As noted by Zytynski et al. [20], the variation of \( E_i \) using equation 13) violates the principle of conservation of energy. Thus, depending on the direction of a closed stress loop, the model will generate or dissipate energy in violation of the basic premise of elastic behavior.


   “They can give satisfactory results for only a limited class of problems … For instance, if we are interested in stress-deformation analysis of a nonlinear semi-infinite medium under a monotonically increasing point or strip load, the results could be acceptable. However, for problems involving loading and unloading, and various stress paths in the soil, results from the hyperbolic simulation may not be reliable, one of the major limitations being that the model includes only one stress path, whereas loading and/or unloading could cause a wide range of stress paths …”
Further details pertaining to the hyperbolic model can be found in the aforementioned references by Duncan and his co-workers, as well as in the report by Wong and Duncan [19]. The use of the model to simulate various boundary-value problems is described in sundry papers [4] and technical reports [5, 16, 17].
Bibliography


