ELEMENT IRREDUCIBLE Q8P0 command

Synopsis

The ELEMENT IRREDUCIBLE Q8P0 command is used to describe all irreducible 8-node quadratic quadrilateral continuum elements that are to be used in mechanical analyses.

Syntax

The following syntax is used to describe a typical Q8P0 irreducible quadrilateral continuum element:

```
ELEMENT IRReducible TYPE Q8P0 NODes #:#:#
  (MATerial #) (INItial #) (THIckness #:#)
    (INTcode #)
  (CONstruction #) (EXCavation #)
  (1_Additional #) (1_Increment #)
  (2_Additional #) (2_Increment #)
    (DONT_PRINT_Results)
  (DONT_PRINT_TRAnst) (DONT_PRINT_STREsses)
    (PRINT_AVG_TRAnst) (PRINT_AVG_STREsses)
  (PRINT_PRIN_TRAnst) (PRINT_PRIN_STREsses)
    (PRINT_VOLUMETRIC_TRAnst) (PRINT_AVG_VOLUMetric_strain)
```
Explanatory Notes

- The **Q8P0** is an irreducible, quadratic, isoparametric “Serendipity” continuum element [1]. The element

  - Contains four (4) vertex nodes.
  - Contains four (4) mid-side nodes.
  - Has two (2) displacements degrees of freedom at each node.
  - Possesses a total of sixteen (16) displacement degrees of freedom.

- The numbering order of NODES associated with **Q8P0** elements, which must be specified sequentially from 1 to 8, is shown in Figure 1.

![Node Numbering](image)

Figure 1: Node Numbering Associated with a Typical Irreducible 8-Node Quadratic Serendipity (Q8P0) Quadrilateral Continuum Element

- The MATERIAL keyword is used to specify the number of the material idealization associated with the element. The default values for the MATERIAL number is one (1).

- The INITIAL keyword is used to specify the initial state number associated with the element. The default value for the INITIAL is zero (0).

- The THICKNESS keyword is used to specify the material thickness assumed for the element. Over a given element, the thickness is assumed to be constant. The default THICKNESS value is equal to one (1.0). For AXISYMMETRIC and PLANE STRAIN idealizations (see discussion of the ANALYSIS IDEALIZATION command), the THICKNESS must be equal to 1.0. For such idealizations, specified values different from 1.0 are ignored and the proper value is used.

- The value specified in conjunction with the INTCODE keyword describes the order of numerical integration scheme to be used in developing the element equations for the element.

The “commonly” used numerical integration rule for **Q8P0** elements corresponds to a 3 by 3 Gauss-Legendre quadrature scheme (degree of precision equal to 5) for the primary dependent variables (i.e., nodal displacements) and a 2 by 2 Gauss-Legendre scheme (degree of precision
equal to 3) for the secondary dependent variables (i.e., strains and stresses). This is the default condition and requires no input using the INTCODE keyword. If a quadrature order different from the default condition is desired, the following integer values are associated with this keyword:

**INTCODE = 32:** a 3 by 3 Gauss-Legendre quadrature scheme (degree of precision equal to 5) is used to compute the primary dependent variables (i.e., nodal displacements) and a 2 by 2 Gauss-Legendre scheme (degree of precision equal to 3) is used to compute the secondary dependent variables (i.e., strains and stresses). This is equivalent to the aforementioned default setting.

**INTCODE = 21:** a 2 by 2 Gauss-Legendre quadrature scheme (degree of precision equal to 3) is used to compute the primary dependent variables (i.e., nodal displacements) and a 1-point Gauss-Legendre scheme (degree of precision equal to 1) is used to compute the secondary dependent variables (i.e., strains and stresses).

- The incremental CONSTRUCTION and EXCAVATION numbers represent the time increment in which the material in this element(s) is added to or removed from the model. A CONSTRUCTION number equal to zero corresponds to a material in existence at the beginning of the analysis. Since this is the default condition, no input is required in such a case. The condition of no excavation is likewise the default.

- If the body being analyzed can be divided into a layer (or layers) of elements, and if the characteristics of the element (i.e., the MATERIAL, the INITIAL state, the incremental CONSTRUCTION and EXCAVATION numbers, the THICKNESS, and the INT-CODE) are the same for several elements within a layer, and if the nodes are numbered in a consistent fashion, then an element data generation option can be employed. To generate a sequence of elements within a single layer, node numbers are specified only for the first element, together with appropriate values for 1_ADDITIONAL and 1_INCREMENT.

If several layers of elements have the same attributes, the above generation option can be carried one step further by making use of the 2_ADDITIONAL, and 2_INCREMENT keywords.

- The purpose of the PRINT commands is to eliminate unnecessary output generated by APES. More precisely, if the time history of strains and/or stresses is desired only for a select few elements, this option greatly speeds program output and facilitates inspection of results by the user. Information associated with the elements specified in this section will be printed for every solution (time) step. If generation is performed using this ELEMENT IRREDUCIBLE command, then all the elements generated will be affected in a like manner by the above print control commands.

- Specification of the keyword DONT_PRINT_Results indicates that the analyst does not desire to see output of secondary dependent variables (i.e., strains and stresses) for this element.

- Specification of the DONT_PRINT_STRAINS keyword indicates that element strains are not to be printed. Under the default condition both strains are printed.
• Specification of the keyword **DONT_PRINT_STRESSES** indicates that stresses are not to be printed. Under the *default* condition stresses are printed.

• The **PRINT_PRIN_STRAINS** keyword indicates that principal strains are to be computed and printed for the element. Under the *default* condition these quantities are not computed and printed.

• The **PRINT_PRIN_STRESSES** keyword indicates that principal stresses are to be computed and printed for the element. Under the *default* condition these quantities are not computed and printed.

• The **PRINT_AVG_STRAINS** keyword indicates that average strains (averaged over the secondary quadrature points) are to be computed and printed for the element. Under the *default* condition average strains are not computed and printed.

• The **PRINT_AVG_STRESSES** keyword indicates that average stresses (averaged over the secondary quadrature points) are to be computed and printed for the element. Under the *default* condition average stresses are not computed and printed.

• The keyword **PRINT_VOLUMETRIC_STRAIN** causes the volumetric strain ($\varepsilon_{\text{vol}}$) to be computed and printed for the element. In addition, the following quantities are computed and printed:

\[
\frac{|\varepsilon_{\text{vol}}|}{\min (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33})} ; \quad \min (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}) \neq 0
\]

\[
\varepsilon_{\text{ratio}} = \frac{|\hat{\varepsilon}_{\text{vol}}^{(e)}|}{\sqrt{\left(\hat{\varepsilon}_{11}^{(e)}\right)^2 + \left(\hat{\varepsilon}_{22}^{(e)}\right)^2 + \left(\hat{\varepsilon}_{33}^{(e)}\right)^2}}
\]

The first quantity is the ratio of the absolute value of $\varepsilon_{\text{vol}}$ to the absolute value of the minimum non-zero normal strain in the element. The second quantity is the ratio of the absolute value of $\varepsilon_{\text{vol}}$ to the square root of the sum of squares of the normal strain components. These ratio are instructive in the assessment of mixed and mixed/penalty elements used to simulate material response in the incompressible limit. As such, this keyword would likely *not* be used in conjunction with the **Q9P0** element. Under the *default* condition the volumetric strain and the aforementioned ratios are *not* computed and printed.

• The keyword **PRINT_AVG_VOLUMETRIC_STRAIN** causes the *average* (over all the quadrature points used to compute secondary dependent variables) volumetric strain to be computed and printed for the element. In addition, the ratio of the absolute value of the *average* volumetric strain to the absolute value of the minimum *average* non-zero normal strain in the element
\[ \varepsilon_{ratio} = \frac{|\varepsilon^{(e)}_{vol}|}{\sqrt{\left(\varepsilon^{(e)}_{11}\right)^2 + \left(\varepsilon^{(e)}_{22}\right)^2 + \left(\varepsilon^{(e)}_{33}\right)^2}} \]

is printed. Under the default condition the average volumetric strain and the aforementioned ratio are not computed and printed.
Examples of Command Usage

Element Performance in Simple “Patch” Test

In order to verify the implementation of the Q8P0 elements, the simple “patch” of elements shown in Figure 2 is analyzed. Only element numbers and the numbers of nodes located along the boundary are shown in this figure. The geometry of the patch is taken from the set of test problems proposed by MacNeal and Harder [2]. Further details pertaining to the so-called engineering patch test are given in Appendix D of [1].

The solution domain is rectangular and measures 0.24 units in the $x$-direction by 0.12 units in the $y$-direction. A plane stress idealization is assumed. Vertical displacement is constrained at nodes 10, 11 and 14. Horizontal displacement is constrained at nodes 10, 13 and 17. A tensile stress equal to 2250.0 is applied along the edge 11-15-12. A compressive stress of 1750.0 is applied along the edge 12-13-16.

![Figure 2: Simple “Patch” of Quadratic (Q8P0) Quadrilateral Elements (only element numbers and numbers of nodes located along boundary are shown)](image)

The input data file associated with the analysis is given below.
ana title "patch test 'C' for quadrilateral mesh"
ana title "after MacNeal & Harder (1985)"

! sig_11 = 2250.0 ; sig_22 = -1750.0 are applied !
echo func off
echo grav off
echo ini off
echo warn off
!
analysis action analyze
analysis type mechanical
analysis idealization plane_stress
analysis temp transient
!
integration time parameter 0.50
!
dim max material isotropic elastic 1
dim max nodes 25
dim max q8p0 = 5
!
finished settings
!
mat elastic isotropic number 1 desc " test 1 " mod = 1.0e+06 poisson 0.25
!
nodes line number 1 x1 0.04 x2 0.02
nodes line number 2 x1 0.18 x2 0.03
nodes line number 3 x1 0.16 x2 0.08
nodes line number 4 x1 0.08 x2 0.08
nodes line number 5 x1 0.110 x2 0.025
nodes line number 6 x1 0.170 x2 0.055
nodes line number 7 x1 0.120 x2 0.080
nodes line number 8 x1 0.060 x2 0.050
nodes line number 9 x1 0.115 x2 0.053
nodes line number 10 x1 0.00 x2 0.00
nodes line number 11 x1 0.24 x2 0.00
nodes line number 12 x1 0.24 x2 0.12
nodes line number 13 x1 0.00 x2 0.12
nodes line number 14 x1 0.120 x2 0.000
nodes line number 15 x1 0.240 x2 0.060
nodes line number 16 x1 0.120 x2 0.120
nodes line number 17 x1 0.000 x2 0.060
nodes line number 18 x1 0.020 x2 0.010
nodes line number 19 x1 0.210 x2 0.015
nodes line number 20 x1 0.200 x2 0.100
nodes line number 21 x1 0.040 x2 0.100
nodes line number 22 x1 0.115 x2 0.013
nodes line number 23 x1 0.205 x2 0.058
nodes line number 24 x1 0.120 x2 0.100
nodes line number 25 x1 0.030 x2 0.055

! element irreducible type "q8p0" nodes 1 10 11 2 18 14 19 5 mat 1 PRINT_AVG_stresses
element irreducible type "q8p0" nodes 2 11 12 3 19 15 20 6 mat 1 PRINT_AVG_stresses
element irreducible type "q8p0" nodes 3 12 13 4 20 16 21 7 mat 1 PRINT_AVG_stresses
element irreducible type "q8p0" nodes 1 14 10 8 21 17 18 5 mat 1 PRINT_AVG_stresses

! specification line quad mech node_b 11 node_end 12 1_incr 1 2_incr 4 1_hist 0 2_hist 0 &
np_begin -2250.0 np_end -2250.0
specification line quad mech node_b 12 node_end 13 1_incr 1 2_incr 4 1_hist 0 2_hist 0 &
np_begin 1750.0 np_end 1750.0
!
specification conc mech nodes 10 1_dis 2_dis
specification conc mech nodes 11 2_dis 1_his 0
specification conc mech nodes 13 1_dis 2_his 0
specification conc mech nodes 14 2_dis
specification conc mech nodes 17 1_dis
!
finished data
!
solution time final 1.0 increments 1 output 1:10:1
!
finished load
The results shown below are obtained using the above data in conjunction with the APES computer program. For clarity, the “header” that is printed at the top of the file is omitted from this file.

patch test 'C' for quadrilateral mesh
after MacNeal & Harder (1985)

======================================================================
| DYNAMIC STORAGE ALLOCATION |
======================================================================

Largest NODE number which can used in the mesh = 25
Max. no. of ISOTROPIC, LINEAR ELASTIC materials = 1
Max. no. of 8-node quad. (Q8P0) elements = 5

======================================================================
= GENERAL ANALYSIS INFORMATION =
======================================================================

--> MECHANICAL analysis shall be performed
--> Fluid flow is NOT accounted for in the analysis
--> Thermal effects are NOT accounted for in analysis

--> TWO-DIMENSIONAL solution domain assumed
   (PLANE STRESS idealization)
--> Nodal coordinates will NOT be updated
--> solver type used: SKYLINE

--> storage type: SYMMETRIC

--> "Isoparametric" scheme used for native mesh generation (if applicable)

======================================================================
= INTEGRATION OPTIONS =
======================================================================

In approximating time derivatives, the value of "THETA" = 5.000E-01
--\ LINEAR analysis

--\ MATERIAL IDEALIZATIONS

-> Material number: 1

\textbf{type : isotropic linear elastic}
\textbf{info. : test 1}

\begin{center}
Modulus of Elasticity = 1.000E+06  
Poisson's ratio = 2.500E-01
\end{center}

\begin{center}
Elastic bulk modulus of the solid phase = 0.000E+00  
Material density of the solid phase = 0.000E+00  
Combined bulk modulus for solid/fluid = 0.000E+00
\end{center}

--\ NODAL COORDINATES

\begin{center}
\begin{tabular}{llll}
node & 1 & x1 = & 4.000E-02  & x2 = 2.000E-02 \\
node & 2 & x1 = & 1.800E-01  & x2 = 3.000E-02 \\
node & 3 & x1 = & 1.600E-01  & x2 = 8.000E-02 \\
node & 4 & x1 = & 8.000E-02  & x2 = 8.000E-02 \\
node & 5 & x1 = & 1.100E-01  & x2 = 2.500E-02 \\
node & 6 & x1 = & 1.700E-01  & x2 = 5.500E-02 \\
node & 7 & x1 = & 1.200E-01  & x2 = 8.000E-02 \\
node & 8 & x1 = & 6.000E-02  & x2 = 5.000E-02 \\
node & 10 & x1 = & 0.000E+00  & x2 = 0.000E+00 \\
node & 11 & x1 = & 2.400E-01  & x2 = 0.000E+00 \\
node & 12 & x1 = & 2.400E-01  & x2 = 1.200E-01
\end{tabular}
\end{center}
node :  13  x1 = 0.000E+00  x2 = 1.200E-01
node :  14  x1 = 1.200E-01  x2 = 0.000E+00
node :  15  x1 = 2.400E-01  x2 = 6.000E-02
node :  16  x1 = 1.200E-01  x2 = 1.200E-01
node :  17  x1 = 0.000E+00  x2 = 6.000E-02
node :  18  x1 = 2.000E-02  x2 = 1.000E-02
node :  19  x1 = 2.100E-01  x2 = 1.500E-02
node :  20  x1 = 2.000E-01  x2 = 1.000E-01
node :  21  x1 = 4.000E-02  x2 = 1.000E-01

======================================================================

=          E  L  E  M  E  N  T   I  N  F  O  R  M  A  T  I  O  N  =
======================================================================

--> number: 1 (type: Q8P0 ) (kind: IRREDUCIBLE )
    nodes:  
             1 10 11 2 18 14 19 5
    integration rule for primary variables: 3 x 3 Gauss-Legendre
    integration rule for secondary variables: 2 x 2 Gauss-Legendre
    material no. = 1
    material type: isotropic linear elastic
    thickness = 1.000E+00

--> number: 2 (type: Q8P0 ) (kind: IRREDUCIBLE )
    nodes:  
             2 11 12 3 19 15 20 6
    integration rule for primary variables: 3 x 3 Gauss-Legendre
    integration rule for secondary variables: 2 x 2 Gauss-Legendre
    material no. = 1
    material type: isotropic linear elastic
    thickness = 1.000E+00

--> number: 3 (type: Q8P0 ) (kind: IRREDUCIBLE )
    nodes:  
             3 12 13 4 20 16 21 7
    integration rule for primary variables: 3 x 3 Gauss-Legendre
    integration rule for secondary variables: 2 x 2 Gauss-Legendre
    material no. = 1
    material type: isotropic linear elastic
    thickness = 1.000E+00

--> number: 4 (type: Q8P0 ) (kind: IRREDUCIBLE )
    nodes:  
             1 4 13 10 8 21 17 18

V. N. Kaliakin
integration rule for primary variables: 3 x 3 Gauss-Legendre
integration rule for secondary variables: 2 x 2 Gauss-Legendre
material no. = 1
material type: isotropic linear elastic
thickness = 1.000E+00

--> number: 5 (type: Q8P0 ) (kind: IRREDUCIBLE )

nodes:
1 2 3 4 5 6 7 8
integration rule for primary variables: 3 x 3 Gauss-Legendre
integration rule for secondary variables: 2 x 2 Gauss-Legendre
material no. = 1
material type: isotropic linear elastic
thickness = 1.000E+00

======================================================================
= N O D E P O I N T S P E C I F I C A T I O N S =
======================================================================

Node Number (coordinates) specification: ~~~~~~~~~~~

10 : ( x1 = 0.000E+00, x2 = 0.000E+00 )
displacement-1 = 0.000E+00 ; history no. = -2
displacement-2 = 0.000E+00 ; history no. = -2

11 : ( x1 = 2.400E-01, x2 = 0.000E+00 )
force-1 = 4.500E+01 ; history no. = 0
force-2 = 0.000E+00 ; history no. = -2

12 : ( x1 = 2.400E-01, x2 = 1.200E-01 )
force-1 = 4.500E+01 ; history no. = 0
force-2 = -7.000E+01 ; history no. = 0

13 : ( x1 = 0.000E+00, x2 = 1.200E-01 )
displacement-1 = 0.000E+00 ; history no. = -2
force-2 = -7.000E+01 ; history no. = 0

14 : ( x1 = 1.200E-01, x2 = 0.000E+00 )
force-1 = 0.000E+00 ; history no. = -2
displacement-2 = 0.000E+00 ; history no. = -2
15 : ( x1 = 2.400E-01, x2 = 6.000E-02 )
force-1 = 1.800E+02 ; history no. = 0
force-2 = -6.939E-15 ; history no. = 0

16 : ( x1 = 1.200E-01, x2 = 1.200E-01 )
force-1 = -2.698E-15 ; history no. = 0
force-2 = -2.800E+02 ; history no. = 0

17 : ( x1 = 0.000E+00, x2 = 6.000E-02 )
displacement-1 = 0.000E+00 ; history no. = -2
force-2 = 0.000E+00 ; history no. = -2

end of mathematical model data

At time 1.000E+00 (step no. 1): NO iteration was required

ELEMENT STRAINS & STRESSES =

--> element 1 ( type = Q8P0 ):

@ ( x1 = 6.560E-02, x2 = 1.744E-02):
eps_11 = 2.688E-03 ; eps_22 = -2.313E-03 ; eps_33 = -1.250E-04 ; gam_12 = -1.764E-10
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -7.056E-05

@ ( x1 = 5.471E-02, x2 = 4.673E-03):
eps_11 = 2.688E-03 ; eps_22 = -2.313E-03 ; eps_33 = -1.250E-04 ; gam_12 = -7.070E-11
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -2.828E-05

@ ( x1 = 1.811E-01, x2 = 5.893E-03):
eps_11 = 2.687E-03 ; eps_22 = -2.312E-03 ; eps_33 = -1.250E-04 ; gam_12 = 3.542E-11
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = 1.417E-05
@ (x1 = 1.586E-01, x2 = 2.199E-02):
eps_11 = 2.687E-03 ; eps_22 = -2.312E-03 ; eps_33 = -1.250E-04 ; gam_12 = 7.208E-11
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = 2.883E-05

average stresses:
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -1.396E-05

--> element 2 ( type = Q8P0 ):

@ (x1 = 1.893E-01, x2 = 3.735E-02):
eps_11 = 2.688E-03 ; eps_22 = -2.312E-03 ; eps_33 = -1.250E-04 ; gam_12 = 6.858E-11
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = 2.743E-05

@ (x1 = 2.264E-01, x2 = 2.857E-02):
eps_11 = 2.687E-03 ; eps_22 = -2.312E-03 ; eps_33 = -1.250E-04 ; gam_12 = 2.074E-10
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = 8.296E-05

@ (x1 = 2.240E-01, x2 = 8.931E-02):
eps_11 = 2.687E-03 ; eps_22 = -2.312E-03 ; eps_33 = -1.250E-04 ; gam_12 = -4.077E-10
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -1.631E-04

@ (x1 = 1.802E-01, x2 = 7.476E-02):
eps_11 = 2.687E-03 ; eps_22 = -2.312E-03 ; eps_33 = -1.250E-04 ; gam_12 = -3.558E-11
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -1.423E-05

average stresses:
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -1.673E-05

--> element 3 ( type = Q8P0 ):

@ (x1 = 1.529E-01, x2 = 8.845E-02):
eps_11 = 2.687E-03 ; eps_22 = -2.312E-03 ; eps_33 = -1.250E-04 ; gam_12 = -1.561E-10
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -6.245E-05

@ (x1 = 1.795E-01, x2 = 1.115E-01):
eps_11 = 2.687E-03 ; eps_22 = -2.312E-03 ; eps_33 = -1.250E-04 ; gam_12 = 3.004E-11
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = 1.202E-05

@ (x1 = 6.048E-02, x2 = 1.115E-01):
eps_11 = 2.687E-03 ; eps_22 = -2.313E-03 ; eps_33 = -1.250E-04 ; gam_12 = -2.314E-10
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -9.256E-05

@(x1 = 8.715E-02, x2 = 8.845E-02):
eps_11 = 2.687E-03 ; eps_22 = -2.313E-03 ; eps_33 = -1.250E-04 ; gam_12 = -7.872E-11
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -3.149E-05

average stresses:
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -4.362E-05

--> element 4 ( type = Q8P0 ):

average stresses:
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -9.078E-05

--> element 5 ( type = Q8P0 ):

average stresses:
sig_11 = 2.250E+03 ; sig_22 = -1.750E+03 ; sig_33 = 0.000E+00 ; sig_12 = -9.078E-05
sig_{11} = 2.250E+03 ; sig_{22} = -1.750E+03 ; sig_{33} = 0.000E+00 ; sig_{12} = -5.563E-05

@ (x1 = 1.446E-01, x2 = 6.899E-02):
sig_{11} = 2.250E+03 ; sig_{22} = -1.750E+03 ; sig_{33} = 0.000E+00 ; sig_{12} = -4.164E-05

eps_{11} = 2.687E-03 ; eps_{22} = -2.312E-03 ; eps_{33} = -1.250E-04 ; gam_{12} = -1.041E-10

@ (x1 = 9.113E-02, x2 = 6.777E-02):
sig_{11} = 2.250E+03 ; sig_{22} = -1.750E+03 ; sig_{33} = 0.000E+00 ; sig_{12} = -4.918E-05

eps_{11} = 2.687E-03 ; eps_{22} = -2.313E-03 ; eps_{33} = -1.250E-04 ; gam_{12} = -1.229E-10

average stresses:
sig_{11} = 2.250E+03 ; sig_{22} = -1.750E+03 ; sig_{33} = 0.000E+00 ; sig_{12} = -5.528E-05

At time 1.000E+00 (step no. 1):

node: 1 ( x1 = 4.000E-02, x2 = 2.000E-02 )
    u_1 = 1.075E-04, u_2 = -4.625E-05

node: 2 ( x1 = 1.800E-01, x2 = 3.000E-02 )
    u_1 = 4.837E-04, u_2 = -6.937E-05

node: 3 ( x1 = 1.600E-01, x2 = 8.000E-02 )
    u_1 = 4.300E-04, u_2 = -1.850E-04

node: 4 ( x1 = 8.000E-02, x2 = 8.000E-02 )
    u_1 = 2.150E-04, u_2 = -1.850E-04

node: 5 ( x1 = 1.100E-01, x2 = 2.500E-02 )
    u_1 = 2.956E-04, u_2 = -5.781E-05

node: 6 ( x1 = 1.700E-01, x2 = 5.500E-02 )
    u_1 = 4.569E-04, u_2 = -1.272E-04

node: 7 ( x1 = 1.200E-01, x2 = 8.000E-02 )
    u_1 = 3.225E-04, u_2 = -1.850E-04
node: 8 ( x1 = 6.000E-02, x2 = 5.000E-02 )
    u_1 = 1.612E-04, u_2 = -1.156E-04

node: 10 ( x1 = 0.000E+00, x2 = 0.000E+00 )
    u_1 = 2.325E-25, u_2 = -1.393E-25

node: 11 ( x1 = 2.400E-01, x2 = 0.000E+00 )
    u_1 = 6.450E-04, u_2 = -2.074E-25

node: 12 ( x1 = 2.400E-01, x2 = 1.200E-01 )
    u_1 = 6.450E-04, u_2 = -2.775E-04

node: 13 ( x1 = 0.000E+00, x2 = 1.200E-01 )
    u_1 = 5.258E-25, u_2 = -2.775E-04

node: 14 ( x1 = 1.200E-01, x2 = 0.000E+00 )
    u_1 = 3.225E-04, u_2 = -4.635E-25

node: 15 ( x1 = 2.400E-01, x2 = 6.000E-02 )
    u_1 = 6.450E-04, u_2 = -1.387E-04

node: 16 ( x1 = 1.200E-01, x2 = 1.200E-01 )
    u_1 = 3.225E-04, u_2 = -2.775E-04

node: 17 ( x1 = 0.000E+00, x2 = 6.000E-02 )
    u_1 = 1.055E-24, u_2 = -1.388E-04

node: 18 ( x1 = 2.000E-02, x2 = 1.000E-02 )
    u_1 = 5.375E-05, u_2 = -2.313E-05

node: 19 ( x1 = 2.100E-01, x2 = 1.500E-02 )
    u_1 = 5.644E-04, u_2 = -3.469E-05

node: 20 ( x1 = 2.000E-01, x2 = 1.000E-01 )
    u_1 = 5.375E-04, u_2 = -2.312E-04

node: 21 ( x1 = 4.000E-02, x2 = 1.000E-01 )
    u_1 = 1.075E-04, u_2 = -2.313E-04

apes -> end of analysis . . . . . . . .
Analysis of a Propped Cantilever Beam

This sample analysis illustrates the ease with which, in the course of an analysis, meshes can be refined. The problem considered is the bending of a propped cantilever beam loaded by a triangularly distributed load as shown in Figure 3.

![Figure 3: Physical Problem Analyzed](image)

The beam is assumed to have a *unit* thickness and a plane stress idealization is adopted. The material is assumed to be isotropic linear elastic with a Poisson’s ratio equal to 0.22 and an elastic modulus equal to $20 \times 10^6$. The first mesh, which is admittedly rather coarse, consists of two bi-quadratic 8-node ($Q_8P_0$) elements of equal size (Figure 4).

![Figure 4: First (coarse) Finite Element Model Used in Analysis](image)

The associated input data file is given below.

```plaintext
analysis title "linear plane stress analysis of a propped cantilever beam"
analysis title "using Q8 irreducible elements"
analysis title "COARSE mesh"
analysis title "note the use of the missing node feature in APES"
!
analysis action analyze ! linear analysis is the default
analysis type mechanical ! plane stress idealization = default
analysis temp transient !

echo initial off
```

V. N. Kaliakin
echo gravity off
echo functions off
!
dim max material isotropic elastic 1
dim max nodes 15
dim max q8p0 2
!
finished settings
!
material elastic isotropic number 1 descrip "beam’s material" &
   modulus 20.0e+06 poissons 0.22
!
nodes line number 1
   ! default coordinates = zero
nodes line number 13 x1 12.0 incr 3
nodes line number 15 x1 12.0 x2 1.5 incr 1
nodes line number 3 x2 1.5 incr -3
nodes line number -1 incr -1
!
element irreducible type "q8p0" nodes 1 7 9 3 4 8 6 2 mat 1 &
   1_add 1 1_incr 6
!
generate surfaces
!
specification line quad mech node_b 15 node_end 3 1_incr -6 2_incr -3 &
   2_hist 0 np_begin 80.0 np_end 0.0
!
spec conc mech nodes 13:15 2_disp 1_forc &
spec conc mech nodes 14 1_disp 2_disp
spec conc mech nodes 1 1_forc 2_disp
!
finished data
!
solution time final 1.0 increments 1 output 1
!
finished load

Using the above data in conjunction with the APES computer program, the results shown below are obtained. For clarity, the “header” that is printed at the top of the file is omitted from this file.

linear plane stress analysis of a propped cantilever beam using Q8 irreducible elements COARSE mesh note the use of the missing node feature in APES
| DYNAMIC STORAGE ALLOCATION |

Largest NODE number which can used in the mesh = 15
Max. no. of ISOTROPIC, LINEAR ELASTIC materials = 1
Max. no. of 8-node quad. (Q8P0) elements = 2

= GENERAL ANALYSIS INFORMATION =

--> MECHANICAL analysis shall be performed
--> Fluid flow is NOT accounted for in the analysis
--> Thermal effects are NOT accounted for in analysis

--> TWO-DIMENSIONAL solution domain assumed
   (PLANE STRESS idealization)

--> Nodal coordinates will NOT be updated

--> solver type used : SKYLINE
--> storage type : SYMMETRIC

--> "Isoparametric" mesh generation scheme used

= INTEGRATION OPTIONS =

In approximating time derivatives, the value of "THETA" = 6.667E-01
NONLINEAR ANALYSIS INFORMATION =
======================================================================

---> LINEAR analysis

======================================================================

MATERIAL IDEALIZATIONS =
======================================================================

---> Material number: 1

--------------
type : isotropic linear elastic
info. : beams material

Modulus of Elasticity = 2.000E+07
Poisson’s ratio = 2.200E-01

Elastic bulk modulus of the solid phase = 0.000E+00
Material density of the solid phase = 0.000E+00
Combined bulk modulus for solid/fluid = 0.000E+00

======================================================================

NODAL COORDINATES =
======================================================================

node : 1  x1 = 3.009E-36  x2 = 7.523E-37
node : 2  x1 = -6.019E-36 x2 = 7.500E-01
node : 3  x1 = -3.009E-36 x2 = 1.500E+00
node : 4  x1 = 3.000E+00  x2 = -9.028E-36
node : 6  x1 = 3.000E+00  x2 = 1.500E+00
node : 7  x1 = 6.000E+00  x2 = -1.053E-35
node : 8  x1 = 6.000E+00  x2 = 7.500E-01
node : 9  x1 = 6.000E+00  x2 = 1.500E+00
node : 10 x1 = 9.000E+00  x2 = -9.028E-36
node : 12 x1 = 9.000E+00  x2 = 1.500E+00
node : 13 x1 = 1.200E+01  x2 = -1.128E-36
node : 14 x1 = 1.200E+01  x2 = 7.500E-01
node : 15 x1 = 1.200E+01  x2 = 1.500E+00
\begin{verbatim}
= ELEMENT INFORMATION =

--> number: 1 (type: Q8P0 ) (kind: IRREDUCIBLE )
     nodes:  1   7    9   3    4    8   6   2
         integration rule for primary variables: 3 x 3 Gauss-Legendre
         integration rule for secondary variables: 2 x 2 Gauss-Legendre
         material no. = 1
         material type: isotropic linear elastic
         thickness = 1.000E+00

--> number: 2 (type: Q8P0 ) (kind: IRREDUCIBLE )
     nodes:  7  13  15   9  10  14  12  8
         integration rule for primary variables: 3 x 3 Gauss-Legendre
         integration rule for secondary variables: 2 x 2 Gauss-Legendre
         material no. = 1
         material type: isotropic linear elastic
         thickness = 1.000E+00

= NODE POINT SPECIFICATIONS =

Node Number (coordinates) specification:

1 : ( x1 = 3.009E-36, x2 = 7.523E-37 )
    force-1 = 0.000E+00 ; history no. = -2
    displacement-2 = 0.000E+00 ; history no. = -2

3 : ( x1 = -3.009E-36, x2 = 1.500E+00 )
    force-1 = 0.000E+00 ; history no. = -2
    force-2 = -2.320E-16 ; history no. = 0

6 : ( x1 = 3.000E+00, x2 = 1.500E+00 )
    force-1 = 0.000E+00 ; history no. = -2
    force-2 = -8.000E+01 ; history no. = 0

9 : ( x1 = 6.000E+00, x2 = 1.500E+00 )
\end{verbatim}
force-1 = 0.000E+00 ; history no. = -2
force-2 = -8.000E+01 ; history no. = 0

12 : ( x1 = 9.000E+00, x2 = 1.500E+00 )
force-1 = 0.000E+00 ; history no. = -2
force-2 = -2.400E+02 ; history no. = 0

13 : ( x1 = 1.200E+01, x2 = -1.128E-36 )
displacement-1 = 0.000E+00 ; history no. = -2
force-2 = 0.000E+00 ; history no. = 0

14 : ( x1 = 1.200E+01, x2 = 7.500E-01 )
displacement-1 = 0.000E+00 ; history no. = -2
displacement-2 = 0.000E+00 ; history no. = -2

15 : ( x1 = 1.200E+01, x2 = 1.500E+00 )
displacement-1 = 0.000E+00 ; history no. = -2
force-2 = -8.000E+01 ; history no. = 0

end of mathematical model data

At time 1.000E+00 (step no. 1) NO iteration was required

--- ELEMENT STRAINS & STRESSES ---

--> element 1 ( type = Q8P0 ):

@ (x1 = 1.268E+00, x2 = 3.170E-01):
eps_11 = 1.447E-05 ; eps_22 = -4.980E-06 ; eps_33 = -2.676E-06 ; eps_12 = -7.345E-06
sig_11 = 2.811E+02 ; sig_22 = -3.776E+01 ; sig_33 = 0.000E+00 ; sig_12 = -6.021E+01

@ (x1 = 4.732E+00, x2 = 3.170E-01):
eps_11 = 2.046E-05 ; eps_22 = -4.069E-06 ; eps_33 = -4.622E-06 ; eps_12 = -1.447E-06
\[ \text{sig}_{11} = 4.111 \times 10^2 \quad ; \quad \text{sig}_{22} = 9.064 \times 10^0 \quad ; \quad \text{sig}_{33} = 0.000 \times 10^0 \quad ; \quad \text{sig}_{12} = -1.186 \times 10^1 \] 

@ (x1 = 4.732 \times 10^0, x2 = 1.183 \times 10^0):
\[ \text{eps}_{11} = -2.031 \times 10^{-5} \quad ; \quad \text{eps}_{22} = 3.425 \times 10^{-6} \quad ; \quad \text{eps}_{33} = 4.764 \times 10^{-6} \quad ; \quad \text{eps}_{12} = -1.186 \times 10^{-6} \] 
\[ \text{sig}_{11} = -4.111 \times 10^2 \quad ; \quad \text{sig}_{22} = -2.195 \times 10^1 \quad ; \quad \text{sig}_{33} = 0.000 \times 10^0 \quad ; \quad \text{sig}_{12} = -1.519 \times 10^1 \] 

@ (x1 = 1.268 \times 10^0, x2 = 1.183 \times 10^0):
\[ \text{eps}_{11} = -1.388 \times 10^{-5} \quad ; \quad \text{eps}_{22} = 2.290 \times 10^{-6} \quad ; \quad \text{eps}_{33} = 3.268 \times 10^{-6} \quad ; \quad \text{eps}_{12} = -7.225 \times 10^{-6} \] 
\[ \text{sig}_{11} = -2.811 \times 10^2 \quad ; \quad \text{sig}_{22} = -1.603 \times 10^1 \quad ; \quad \text{sig}_{33} = 0.000 \times 10^0 \quad ; \quad \text{sig}_{12} = -5.922 \times 10^1 \] 

--- Element 2 (type = Q8P0) ---

\[ \text{eps}_{11} = 7.285 \times 10^{-6} \quad ; \quad \text{eps}_{22} = -2.696 \times 10^{-6} \quad ; \quad \text{eps}_{33} = -1.294 \times 10^{-6} \quad ; \quad \text{eps}_{12} = 6.405 \times 10^{-6} \] 

@ (x1 = 7.268 \times 10^0, x2 = 3.170 \times 10^{-1}):
\[ \text{sig}_{11} = 1.406 \times 10^2 \quad ; \quad \text{sig}_{22} = -2.298 \times 10^1 \quad ; \quad \text{sig}_{33} = 0.000 \times 10^0 \quad ; \quad \text{sig}_{12} = 5.250 \times 10^1 \] 

@ (x1 = 1.073 \times 10^1, x2 = 3.170 \times 10^{-1}):
\[ \text{eps}_{11} = -1.418 \times 10^{-5} \quad ; \quad \text{eps}_{22} = 3.664 \times 10^{-6} \quad ; \quad \text{eps}_{33} = 2.967 \times 10^{-6} \quad ; \quad \text{eps}_{12} = 2.356 \times 10^{-5} \] 
\[ \text{sig}_{11} = -2.811 \times 10^2 \quad ; \quad \text{sig}_{22} = 1.143 \times 10^1 \quad ; \quad \text{sig}_{33} = 0.000 \times 10^0 \quad ; \quad \text{sig}_{12} = 1.931 \times 10^2 \] 

@ (x1 = 1.073 \times 10^1, x2 = 1.183 \times 10^0):
\[ \text{eps}_{11} = 1.495 \times 10^{-5} \quad ; \quad \text{eps}_{22} = -7.165 \times 10^{-6} \quad ; \quad \text{eps}_{33} = -2.197 \times 10^{-6} \quad ; \quad \text{eps}_{12} = 2.345 \times 10^{-5} \] 
\[ \text{sig}_{11} = 2.811 \times 10^2 \quad ; \quad \text{sig}_{22} = -8.144 \times 10^1 \quad ; \quad \text{sig}_{33} = 0.000 \times 10^0 \quad ; \quad \text{sig}_{12} = 1.922 \times 10^2 \] 

@ (x1 = 7.268 \times 10^0, x2 = 1.183 \times 10^0):
\[ \text{eps}_{11} = -6.680 \times 10^{-6} \quad ; \quad \text{eps}_{22} = -5.182 \times 10^{-8} \quad ; \quad \text{eps}_{33} = 1.899 \times 10^{-6} \quad ; \quad \text{eps}_{12} = 6.795 \times 10^{-6} \] 
\[ \text{sig}_{11} = -1.406 \times 10^2 \quad ; \quad \text{sig}_{22} = -3.198 \times 10^1 \quad ; \quad \text{sig}_{33} = 0.000 \times 10^0 \quad ; \quad \text{sig}_{12} = 5.569 \times 10^1 \] 

Maximum values of element variables:

| max | eps_{11} | = 2.046 \times 10^{-5} | x1 = 4.732 \times 10^0, x2 = 3.170 \times 10^{-1} |
| max | eps_{22} | = 7.165 \times 10^{-6} | x1 = 1.073 \times 10^1, x2 = 1.183 \times 10^0 |
| max | eps_{33} | = 4.764 \times 10^{-6} | x1 = 4.732 \times 10^0, x2 = 1.183 \times 10^0 |
| max | eps_{12} | = 2.356 \times 10^{-5} | x1 = 1.073 \times 10^1, x2 = 1.183 \times 10^0 |
| max | sig_{11} | = 4.111 \times 10^2 | x1 = 4.732 \times 10^0, x2 = 1.183 \times 10^0 |
| max | sig_{22} | = 8.144 \times 10^1 | x1 = 1.073 \times 10^1, x2 = 1.183 \times 10^0 |
| max | sig_{33} | = 0.000 \times 10^0 | x1 = 1.268 \times 10^0, x2 = 3.170 \times 10^{-1} |
| max | sig_{12} | = 1.931 \times 10^2 | x1 = 1.073 \times 10^1, x2 = 3.170 \times 10^{-1} |
node :  1 ( x1 = 3.009E-36, x2 = 7.523E-37 )  
       u_1 = -1.436E-04, u_2 = -1.908E-26

node :  2 ( x1 = -6.019E-36, x2 = 7.500E-01 )  
       u_1 = -3.026E-06, u_2 = -4.765E-06

node :  3 ( x1 = -3.009E-36, x2 = 1.500E+00 )  
       u_1 = 1.367E-04, u_2 = -4.862E-06

node :  4 ( x1 = 3.000E+00, x2 = -9.028E-36 )  
       u_1 = -6.675E-05, u_2 = -4.414E-04

node :  6 ( x1 = 3.000E+00, x2 = 1.500E+00 )  
       u_1 = 6.194E-05, u_2 = -4.415E-04

node :  7 ( x1 = 6.000E+00, x2 = -1.053E-35 )  
       u_1 = 3.745E-05, u_2 = -5.055E-04

node :  8 ( x1 = 6.000E+00, x2 = 7.500E-01 )  
       u_1 = -2.112E-06, u_2 = -5.091E-04

node :  9 ( x1 = 6.000E+00, x2 = 1.500E+00 )  
       u_1 = -4.138E-05, u_2 = -5.077E-04

node :  10 ( x1 = 9.000E+00, x2 = -9.028E-36 )  
       u_1 = 6.711E-05, u_2 = -2.373E-04

node :  12 ( x1 = 9.000E+00, x2 = 1.500E+00 )  
       u_1 = -2.368E-25, u_2 = -6.710E-06

node :  13 ( x1 = 1.200E+01, x2 = -1.128E-36 )  
       u_1 = 2.369E-25, u_2 = -3.526E-06

node :  14 ( x1 = 1.200E+01, x2 = 7.500E-01 )  
       u_1 = -4.392E-29, u_2 = -2.558E-26

node :  15 ( x1 = 1.200E+01, x2 = 1.500E+00 )  
       u_1 = -2.368E-25, u_2 = -6.710E-06
The mesh is next refined into eight elements. The associated mathematical model is shown in Figure 5.

![Figure 5: Second (refined) Finite Element Model Used in Analysis](image)

The associated input data file is shown below. Note that the mesh refinement was realized by simply changing a few values in the input data file.

```plaintext
analysis title "linear plane stress analysis of a propped cantilever beam"
analysis title "using Q8 irreducible elements"
analysis title "REFINED mesh of 8 elements"
analysis title "note the use of the missing node feature in APES"
!
analysis action analyze ! linear analysis is the default
analysis type mechanical ! plane stress idealization = default
analysis temp transient
!
echo initial off
echo gravity off
echo functions off
!
dim max material isotropic elastic 1

dim max nodes 45
dim max q8p0 8
!
finished settings
!
material elastic isotropic number 1 descrip "beam’s material" &
  modulus  20.0e+06  poissons  0.22
!
nodes line number  1
nodes line number  41  x1  12.0 incr  5
```
Using the above data, the results shown below are obtained. For clarity, the “header” that is printed at the top of the file is omitted from this file.

linear plane stress analysis of a propped cantilever beam using Q8 irreducible elements REFINED mesh of 8 elements note the use of the missing node feature in APES

======================================================================
| DYNAMIC STORAGE ALLOCATION |
======================================================================

Largest NODE number which can used in the mesh = 45

Max. no. of ISOTROPIC, LINEAR ELASTIC materials = 1

Max. no. of 8-node quad. (Q8P0) elements = 8

======================================================================
= GENERAL ANALYSIS INFORMATION =
======================================================================
MECHANICAL analysis shall be performed
Fluid flow is NOT accounted for in the analysis
Thermal effects are NOT accounted for in analysis

TWO-DIMENSIONAL solution domain assumed
(PLANE STRESS idealization)

Nodal coordinates will NOT be updated

solver type used: SKYLINE
storage type: SYMMETRIC

"Isoparametric" mesh generation scheme used

Integration Options

In approximating time derivatives, the value of "THETA" = \(6.667 \times 10^{-1}\)

Nonlinear Analysis Information

LINEAR analysis

Material Idealizations

Material number: 1

type: isotropic linear elastic
info. : plate material

Modulus of Elasticity = 2.000E+07
Poisson’s ratio = 2.200E-01

Elastic bulk modulus of the solid phase = 0.000E+00
Material density of the solid phase = 0.000E+00
Combined bulk modulus for solid/fluid = 0.000E+00

======================================================================
= N O D A L C O O R D I N A T E S =
======================================================================

node : 1 x1 = -3.461E-35 x2 = -5.642E-36
node : 2 x1 = 9.028E-36 x2 = 3.750E-01
node : 3 x1 = 5.868E-35 x2 = 7.500E-01
node : 4 x1 = 7.041E-36 x2 = 1.125E+00
node : 5 x1 = -3.259E-35 x2 = 1.500E+00
node : 6 x1 = 1.500E+00 x2 = -1.652E-36
node : 8 x1 = 1.500E+00 x2 = 7.500E-01
node : 10 x1 = 1.500E+00 x2 = 1.500E+00
node : 11 x1 = 3.000E+00 x2 = 3.616E-36
node : 12 x1 = 3.000E+00 x2 = 3.750E-01
node : 13 x1 = 3.000E+00 x2 = 7.500E-01
node : 14 x1 = 3.000E+00 x2 = 1.125E+00
node : 15 x1 = 3.000E+00 x2 = 1.500E+00
node : 16 x1 = 4.500E+00 x2 = 8.784E-36
node : 18 x1 = 4.500E+00 x2 = 7.500E-01
node : 20 x1 = 4.500E+00 x2 = 1.500E+00
node : 21 x1 = 6.000E+00 x2 = 5.143E-38
node : 22 x1 = 6.000E+00 x2 = 3.750E-01
node : 23 x1 = 6.000E+00 x2 = 7.500E-01
node : 24 x1 = 6.000E+00 x2 = 1.125E+00
node : 25 x1 = 6.000E+00 x2 = 1.500E+00
node : 26 x1 = 7.500E+00 x2 = 2.383E-36
node : 28 x1 = 7.500E+00 x2 = 7.500E-01
node : 30 x1 = 7.500E+00 x2 = 1.500E+00
node : 31 x1 = 9.000E+00 x2 = 2.961E-36
node : 32 x1 = 9.000E+00 x2 = 3.750E-01
node : 33 x1 = 9.000E+00 x2 = 7.500E-01
node : 34 x1 = 9.000E+00 x2 = 1.125E+00
node : 35 x1 = 9.000E+00 x2 = 1.500E+00
node : 36 x1 = 1.050E+01 x2 = -6.019E-36
node :  38  x1 =  1.050E+01  x2 =  7.500E-01
node :  40  x1 =  1.050E+01  x2 =  1.500E+00
node :  41  x1 =  1.200E+01  x2 =  -2.069E-36
node :  42  x1 =  1.200E+01  x2 =  3.750E-01
node :  43  x1 =  1.200E+01  x2 =  7.500E-01
node :  44  x1 =  1.200E+01  x2 =  1.125E+00
node :  45  x1 =  1.200E+01  x2 =  1.500E+00

======================================================================

=  E L E M E N T  I N F O R M A T I O N  =

======================================================================

--> number:  1  (type: Q8P0  )  (kind: IRREDUCIBLE  )
   ~~~~~ nodes:  1  11  13  3  6  12  8  2
            integration rule for primary variables:  3 x 3 Gauss-Legendre
            integration rule for secondary variables:  2 x 2 Gauss-Legendre
            material no. =  1
            material type: isotropic linear elastic
            thickness =  1.000E+00

--> number:  2  (type: Q8P0  )  (kind: IRREDUCIBLE  )
   ~~~~~ nodes:  11  21  23  13  16  22  18  12
            integration rule for primary variables:  3 x 3 Gauss-Legendre
            integration rule for secondary variables:  2 x 2 Gauss-Legendre
            material no. =  1
            material type: isotropic linear elastic
            thickness =  1.000E+00

--> number:  3  (type: Q8P0  )  (kind: IRREDUCIBLE  )
   ~~~~~ nodes:  21  31  33  23  26  32  28  22
            integration rule for primary variables:  3 x 3 Gauss-Legendre
            integration rule for secondary variables:  2 x 2 Gauss-Legendre
            material no. =  1
            material type: isotropic linear elastic
            thickness =  1.000E+00

--> number:  4  (type: Q8P0  )  (kind: IRREDUCIBLE  )
   ~~~~~ nodes:  31  41  43  33  36  42  38  32
            integration rule for primary variables:  3 x 3 Gauss-Legendre
            integration rule for secondary variables:  2 x 2 Gauss-Legendre
            material no. =  1
material type: isotropic linear elastic

thickness = 1.000E+00

---> number: 5 (type: Q8P0 ) (kind: IRREDUCIBLE )
-----
    nodes:  3  13  15  5  8  14  10  4

integration rule for primary variables: 3 x 3 Gauss-Legendre
integration rule for secondary variables: 2 x 2 Gauss-Legendre
material no. = 1
material type: isotropic linear elastic
thickness = 1.000E+00

---> number: 6 (type: Q8P0 ) (kind: IRREDUCIBLE )
-----
    nodes: 13  23  25  15  18  24  20  14

integration rule for primary variables: 3 x 3 Gauss-Legendre
integration rule for secondary variables: 2 x 2 Gauss-Legendre
material no. = 1
material type: isotropic linear elastic
thickness = 1.000E+00

---> number: 7 (type: Q8P0 ) (kind: IRREDUCIBLE )
-----
    nodes:  23  33  35  25  28  34  30  24

integration rule for primary variables: 3 x 3 Gauss-Legendre
integration rule for secondary variables: 2 x 2 Gauss-Legendre
material no. = 1
material type: isotropic linear elastic
thickness = 1.000E+00

---> number: 8 (type: Q8P0 ) (kind: IRREDUCIBLE )
-----
    nodes:  33  43  45  35  38  44  40  34

integration rule for primary variables: 3 x 3 Gauss-Legendre
integration rule for secondary variables: 2 x 2 Gauss-Legendre
material no. = 1
material type: isotropic linear elastic
thickness = 1.000E+00

---------------------------------------------------------------------
= NODE POINT SPECIFICATIONS =
---------------------------------------------------------------------
### Node (coordinates)

<table>
<thead>
<tr>
<th>Number</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>--------------</td>
</tr>
<tr>
<td>1</td>
<td>( x_1 = -3.461\times 10^{-35}, \quad x_2 = -5.642\times 10^{-36} )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{displacement-2} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td>5</td>
<td>( x_1 = -3.259\times 10^{-35}, \quad x_2 = 1.500E+00 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = -5.800E-17; \quad \text{history no.} = 0 )</td>
</tr>
<tr>
<td>10</td>
<td>( x_1 = 1.500E+00, \quad x_2 = 1.500E+00 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = -2.000E+01; \quad \text{history no.} = 0 )</td>
</tr>
<tr>
<td>15</td>
<td>( x_1 = 3.000E+00, \quad x_2 = 1.500E+00 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = -2.000E+01; \quad \text{history no.} = 0 )</td>
</tr>
<tr>
<td>20</td>
<td>( x_1 = 4.500E+00, \quad x_2 = 1.500E+00 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = -6.000E+01; \quad \text{history no.} = 0 )</td>
</tr>
<tr>
<td>25</td>
<td>( x_1 = 6.000E+00, \quad x_2 = 1.500E+00 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = -4.000E+01; \quad \text{history no.} = 0 )</td>
</tr>
<tr>
<td>30</td>
<td>( x_1 = 7.500E+00, \quad x_2 = 1.500E+00 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = -1.000E+02; \quad \text{history no.} = 0 )</td>
</tr>
<tr>
<td>35</td>
<td>( x_1 = 9.000E+00, \quad x_2 = 1.500E+00 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = -6.000E+01; \quad \text{history no.} = 0 )</td>
</tr>
<tr>
<td>40</td>
<td>( x_1 = 1.050E+01, \quad x_2 = 1.500E+00 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = -1.400E+02; \quad \text{history no.} = 0 )</td>
</tr>
<tr>
<td>41</td>
<td>( x_1 = 1.200E+01, \quad x_2 = -2.069E-36 )</td>
</tr>
<tr>
<td></td>
<td>( \text{displacement-1} = 0.000E+00; \quad \text{history no.} = -2 )</td>
</tr>
<tr>
<td></td>
<td>( \text{force-2} = 0.000E+00; \quad \text{history no.} = 0 )</td>
</tr>
</tbody>
</table>
At time 1.000E+00 (step no. 1) NO iteration was required

---

\[
\begin{align*}
42 & : ( x_1 = 1.200E+01, x_2 = 3.750E-01 ) \\
 & \quad \text{displacement-1} = 0.000E+00 \ ; \ \text{history no.} = -2 \\
 & \quad \text{force-2} = 0.000E+00 \ ; \ \text{history no.} = 0 \\
43 & : ( x_1 = 1.200E+01, x_2 = 7.500E-01 ) \\
 & \quad \text{displacement-1} = 0.000E+00 \ ; \ \text{history no.} = -2 \\
 & \quad \text{displacement-2} = 0.000E+00 \ ; \ \text{history no.} = -2 \\
44 & : ( x_1 = 1.200E+01, x_2 = 1.125E+00 ) \\
 & \quad \text{displacement-1} = 0.000E+00 \ ; \ \text{history no.} = -2 \\
 & \quad \text{force-2} = 0.000E+00 \ ; \ \text{history no.} = 0 \\
45 & : ( x_1 = 1.200E+01, x_2 = 1.500E+00 ) \\
 & \quad \text{displacement-1} = 0.000E+00 \ ; \ \text{history no.} = -2 \\
 & \quad \text{force-2} = -4.000E+01 \ ; \ \text{history no.} = 0 \\
\end{align*}
\]

---

end of mathematical model data

---

\[
\begin{align*}
\text{At time 1.000E+00 (step no. 1) NO iteration was required}
\end{align*}
\]

---

\[
\begin{align*}
\text{--- E L E M E N T \ S T R A I N S \ & \ S T R E S S E S ---}
\end{align*}
\]

---

\[
\begin{align*}
\text{--> element 1 ( type = Q8P0 )}:
\end{align*}
\]

\[
\begin{align*}
\text{@}(x_1 = 6.340E-01, x_2 = 1.585E-01): \\
\text{eps}_{11} = 9.760E-06 \ ; \ \text{eps}_{22} = -6.244E-06 \ ; \ \text{eps}_{33} = -9.917E-07 \ ; \ \text{eps}_{12} = -4.808E-06 \\
\text{sig}_{11} = 1.763E+02 \ ; \ \text{sig}_{22} = -8.611E+01 \ ; \ \text{sig}_{33} = 0.000E+00 \ ; \ \text{sig}_{12} = -3.941E+01 \\
\text{@}(x_1 = 2.366E+00, x_2 = 1.585E-01): \\
\text{eps}_{11} = 2.038E-05 \ ; \ \text{eps}_{22} = -3.237E-06 \ ; \ \text{eps}_{33} = -4.836E-06 \ ; \ \text{eps}_{12} = -2.646E-06 \\
\text{sig}_{11} = 4.135E+02 \ ; \ \text{sig}_{22} = 2.621E+01 \ ; \ \text{sig}_{33} = 0.000E+00 \ ; \ \text{sig}_{12} = -2.169E+01 \\
\text{@}(x_1 = 2.366E+00, x_2 = 5.915E-01): \\
\end{align*}
\]
\[ \begin{align*}
\varepsilon_{11} &= 5.053 \times 10^{-6} ;
\varepsilon_{22} &= 1.334 \times 10^{-7} ;
\varepsilon_{33} &= -1.463 \times 10^{-6} ;
\varepsilon_{12} &= -9.179 \times 10^{-6} \\
\sigma_{11} &= 1.068 \times 10^{2} ;
\sigma_{22} &= 2.617 \times 10^{1} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= -7.523 \times 10^{1} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 2.485 \times 10^{-6} ;
\varepsilon_{22} &= -3.650 \times 10^{-6} ;
\varepsilon_{33} &= 3.283 \times 10^{-7} ;
\varepsilon_{12} &= -1.117 \times 10^{-5} \\
\sigma_{11} &= 3.536 \times 10^{1} ;
\sigma_{22} &= -6.521 \times 10^{1} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= -9.157 \times 10^{1} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 3.215 \times 10^{-5} ;
\varepsilon_{22} &= -7.287 \times 10^{-6} ;
\varepsilon_{33} &= -7.014 \times 10^{-6} ;
\varepsilon_{12} &= -3.334 \times 10^{-6} \\
\sigma_{11} &= 6.421 \times 10^{2} ;
\sigma_{22} &= -4.486 \times 10^{0} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= -2.733 \times 10^{1} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 3.590 \times 10^{-5} ;
\varepsilon_{22} &= -7.963 \times 10^{-6} ;
\varepsilon_{33} &= -7.879 \times 10^{-6} ;
\varepsilon_{12} &= 6.462 \times 10^{-6} \\
\sigma_{11} &= 7.177 \times 10^{2} ;
\sigma_{22} &= -1.369 \times 10^{0} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= 5.297 \times 10^{0} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 9.614 \times 10^{-6} ;
\varepsilon_{22} &= -2.207 \times 10^{-6} ;
\varepsilon_{33} &= -2.089 \times 10^{-6} ;
\varepsilon_{12} &= -5.871 \times 10^{-6} \\
\sigma_{11} &= 1.918 \times 10^{2} ;
\sigma_{22} &= -1.930 \times 10^{0} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= -4.498 \times 10^{0} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 8.788 \times 10^{-6} ;
\varepsilon_{22} &= -3.070 \times 10^{-6} ;
\varepsilon_{33} &= -1.613 \times 10^{-6} ;
\varepsilon_{12} &= 3.907 \times 10^{-6} \\
\sigma_{11} &= 1.705 \times 10^{2} ;
\sigma_{22} &= -2.388 \times 10^{1} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= -4.812 \times 10^{0} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 3.156 \times 10^{-5} ;
\varepsilon_{22} &= -7.287 \times 10^{-6} ;
\varepsilon_{33} &= -7.014 \times 10^{-6} ;
\varepsilon_{12} &= 2.369 \times 10^{-6} \\
\sigma_{11} &= 6.307 \times 10^{2} ;
\sigma_{22} &= -1.015 \times 10^{1} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= 1.942 \times 10^{1} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 3.165 \times 10^{-5} ;
\varepsilon_{22} &= -7.963 \times 10^{-6} ;
\varepsilon_{33} &= -7.879 \times 10^{-6} ;
\varepsilon_{12} &= 6.462 \times 10^{-7} \\
\sigma_{11} &= 3.788 \times 10^{2} ;
\sigma_{22} &= 3.756 \times 10^{1} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= 3.202 \times 10^{1} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 4.725 \times 10^{-6} ;
\varepsilon_{22} &= -1.918 \times 10^{-6} ;
\varepsilon_{33} &= -7.917 \times 10^{-7} ;
\varepsilon_{12} &= 1.832 \times 10^{-5} \\
\sigma_{11} &= 9.043 \times 10^{1} ;
\sigma_{22} &= -1.846 \times 10^{1} ;
\sigma_{33} &= 0.000 \times 10^{0} ;
\sigma_{12} &= 1.502 \times 10^{2} \\
\end{align*} \]

\[ \begin{align*}
\varepsilon_{11} &= 8.228 \times 10^{-6} ;
\varepsilon_{22} &= -2.720 \times 10^{-6} ;
\varepsilon_{33} &= -1.554 \times 10^{-6} ;
\varepsilon_{12} &= 5.639 \times 10^{-6} \\
\end{align*} \]
\[ \text{sig}_{11} = 1.604 \times 10^2 ; \text{sig}_{22} = -1.911 \times 10^1 ; \text{sig}_{33} = 0.000 \times 10^0 ; \text{sig}_{12} = 4.622 \times 10^1 \]

--> element 4 (type = Q8P0):

\[ \begin{array}{c}
\text{eps}_{11} = -1.194 \times 10^{-5} ; \text{eps}_{22} = -1.117 \times 10^{-7} ; \text{eps}_{33} = 3.400 \times 10^{-6} ; \text{eps}_{12} = 1.057 \times 10^{-5} \\
\text{sig}_{11} = -2.515 \times 10^2 ; \text{sig}_{22} = -5.757 \times 10^1 ; \text{sig}_{33} = 0.000 \times 10^0 ; \text{sig}_{12} = 8.663 \times 10^1 \\
\end{array} \]

\[ \begin{array}{c}
\text{eps}_{11} = 1.94 \times 10^{-5} ; \text{eps}_{22} = 1.259 \times 10^{-5} ; \text{eps}_{33} = 1.067 \times 10^{-5} ; \text{eps}_{12} = 2.263 \times 10^{-5} \\
\text{sig}_{11} = -1.137 \times 10^2 ; \text{sig}_{22} = 3.134 \times 10^1 ; \text{sig}_{33} = 0.000 \times 10^0 ; \text{sig}_{12} = 1.855 \times 10^2 \\
\end{array} \]

\[ \begin{array}{c}
\text{eps}_{11} = -1.275 \times 10^{-5} ; \text{eps}_{22} = 7.521 \times 10^{-6} ; \text{eps}_{33} = 1.476 \times 10^{-5} ; \text{eps}_{12} = 2.599 \times 10^{-5} \\
\text{sig}_{11} = -6.340 \times 10^{-1} ; \text{sig}_{22} = 9.910 \times 10^{-5} ; \text{sig}_{33} = 0.000 \times 10^0 ; \text{sig}_{12} = 1.966 \times 10^{-1} \\
\end{array} \]

--> element 5 (type = Q8P0):

\[ \begin{array}{c}
\text{eps}_{11} = -2.029 \times 10^{-6} ; \text{eps}_{22} = -1.092 \times 10^{-6} ; \text{eps}_{33} = 8.802 \times 10^{-7} ; \text{eps}_{12} = -1.095 \times 10^{-5} \\
\text{sig}_{11} = 4.769 \times 10^1 ; \text{sig}_{22} = -3.233 \times 10^1 ; \text{sig}_{33} = 0.000 \times 10^0 ; \text{sig}_{12} = -8.972 \times 10^0 \\
\end{array} \]

\[ \begin{array}{c}
\text{eps}_{11} = -5.593 \times 10^{-6} ; \text{eps}_{22} = 1.035 \times 10^{-5} ; \text{eps}_{33} = 1.286 \times 10^{-6} ; \text{eps}_{12} = -9.344 \times 10^{-6} \\
\text{sig}_{11} = 1.128 \times 10^2 ; \text{sig}_{22} = 4.108 \times 10^0 ; \text{sig}_{33} = 0.000 \times 10^0 ; \text{sig}_{12} = -7.659 \times 10^1 \\
\end{array} \]

\[ \begin{array}{c}
\text{eps}_{11} = -2.034 \times 10^{-5} ; \text{eps}_{22} = 4.307 \times 10^{-6} ; \text{eps}_{33} = 4.521 \times 10^{-6} ; \text{eps}_{12} = -4.268 \times 10^{-6} \\
\text{sig}_{11} = 4.075 \times 10^2 ; \text{sig}_{22} = -3.518 \times 10^0 ; \text{sig}_{33} = 0.000 \times 10^0 ; \text{sig}_{12} = -3.498 \times 10^1 \\
\end{array} \]

--> element 6 (type = Q8P0):

\[ \begin{array}{c}
\text{eps}_{11} = -8.077 \times 10^{-6} ; \text{eps}_{22} = 1.257 \times 10^{-6} ; \text{eps}_{33} = 1.923 \times 10^{-6} ; \text{eps}_{12} = -4.147 \times 10^{-6} \\
\text{sig}_{11} = 1.639 \times 10^2 ; \text{sig}_{22} = 1.092 \times 10^1 ; \text{sig}_{33} = 0.000 \times 10^0 ; \text{sig}_{12} = -3.399 \times 10^0 \\
\end{array} \]

35

V. N. Kaliakin
\[ \varepsilon_{11} = -8.212 \times 10^{-6}; \quad \varepsilon_{22} = 1.266 \times 10^{-6}; \quad \varepsilon_{33} = 1.959 \times 10^{-6}; \quad \varepsilon_{12} = -5.917 \times 10^{-6} \]
\[ \sigma_{11} = -1.667 \times 10^{2}; \quad \sigma_{22} = -1.136 \times 10^{1}; \quad \sigma_{33} = 0.000 \times 10^{0}; \quad \sigma_{12} = -4.850 \times 10^{1} \]

\[ \varepsilon_{11} = -9.335 \times 10^{-6}; \quad \varepsilon_{22} = 5.697 \times 10^{-7}; \quad \varepsilon_{33} = 2.472 \times 10^{-6}; \quad \varepsilon_{12} = -7.295 \times 10^{-7} \]
\[ \sigma_{11} = -1.936 \times 10^{2}; \quad \sigma_{22} = -3.119 \times 10^{1}; \quad \sigma_{33} = 0.000 \times 10^{0}; \quad \sigma_{12} = -5.979 \times 10^{0} \]

\[ \varepsilon_{11} = -3.546 \times 10^{-5}; \quad \varepsilon_{22} = 6.336 \times 10^{-6}; \quad \varepsilon_{33} = 8.214 \times 10^{-6}; \quad \varepsilon_{12} = 3.495 \times 10^{-7} \]
\[ \sigma_{11} = -7.160 \times 10^{2}; \quad \sigma_{22} = -3.079 \times 10^{1}; \quad \sigma_{33} = 0.000 \times 10^{0}; \quad \sigma_{12} = 2.864 \times 10^{0} \]

\[ \varepsilon_{11} = -3.193 \times 10^{-5}; \quad \varepsilon_{22} = 5.453 \times 10^{-6}; \quad \varepsilon_{33} = 7.468 \times 10^{-6}; \quad \varepsilon_{12} = -2.065 \times 10^{-6} \]
\[ \sigma_{11} = -6.459 \times 10^{2}; \quad \sigma_{22} = -3.303 \times 10^{1}; \quad \sigma_{33} = 0.000 \times 10^{0}; \quad \sigma_{12} = -1.693 \times 10^{1} \]

--> element 7 ( type = Q8P0 ):

\[ \varepsilon_{11} = -7.695 \times 10^{-6}; \quad \varepsilon_{22} = 4.535 \times 10^{-7}; \quad \varepsilon_{33} = 2.042 \times 10^{-6}; \quad \varepsilon_{12} = 2.397 \times 10^{-5} \]
\[ \sigma_{11} = 6.346 \times 10^{1}; \quad \sigma_{22} = -3.208 \times 10^{0}; \quad \sigma_{33} = 0.000 \times 10^{0}; \quad \sigma_{12} = 1.965 \times 10^{2} \]

\[ \varepsilon_{11} = -4.126 \times 10^{-6}; \quad \varepsilon_{22} = -8.525 \times 10^{-7}; \quad \varepsilon_{33} = 1.404 \times 10^{-6}; \quad \varepsilon_{12} = 1.828 \times 10^{-5} \]
\[ \sigma_{11} = -3.786 \times 10^{2}; \quad \sigma_{22} = -9.286 \times 10^{1}; \quad \sigma_{33} = 0.000 \times 10^{0}; \quad \sigma_{12} = 3.183 \times 10^{1} \]

\[ \varepsilon_{11} = -3.119 \times 10^{-5}; \quad \varepsilon_{22} = 5.198 \times 10^{-6}; \quad \varepsilon_{33} = 7.330 \times 10^{-6}; \quad \varepsilon_{12} = 2.356 \times 10^{-6} \]
\[ \sigma_{11} = 6.346 \times 10^{1}; \quad \sigma_{22} = -3.208 \times 10^{0}; \quad \sigma_{33} = 0.000 \times 10^{0}; \quad \sigma_{12} = 1.965 \times 10^{2} \]

--> element 8 ( type = Q8P0 ):

\[ \varepsilon_{11} = 3.209 \times 10^{-6}; \quad \varepsilon_{22} = -8.585 \times 10^{-7}; \quad \varepsilon_{33} = -6.628 \times 10^{-7}; \quad \varepsilon_{12} = 2.397 \times 10^{-5} \]
\[ \sigma_{11} = 6.346 \times 10^{1}; \quad \sigma_{22} = -3.208 \times 10^{0}; \quad \sigma_{33} = 0.000 \times 10^{0}; \quad \sigma_{12} = 1.965 \times 10^{2} \]

36

V. N. Kaliakin
@ (x1 = 1.137E+01, x2 = 9.085E-01):
eps_11 = 1.361E-05 ; eps_22 = -1.131E-05 ; eps_33 = -6.481E-07 ; eps_12 = 3.170E-05
sig_11 = 2.337E+02 ; sig_22 = -1.748E+02 ; sig_33 = 0.000E+00 ; sig_12 = 2.598E+02

@ (x1 = 1.137E+01, x2 = 1.342E+00):
eps_11 = 5.124E-05 ; eps_22 = -1.637E-05 ; eps_33 = -9.837E-06 ; eps_12 = 2.255E-05
sig_11 = 1.001E+03 ; sig_22 = -1.070E+02 ; sig_33 = 0.000E+00 ; sig_12 = 1.848E+02

@ (x1 = 9.634E+00, x2 = 1.342E+00):
eps_11 = 1.265E-05 ; eps_22 = -3.111E-06 ; eps_33 = -2.690E-06 ; eps_12 = 1.062E-05
sig_11 = 2.515E+02 ; sig_22 = -6.898E+00 ; sig_33 = 0.000E+00 ; sig_12 = 8.702E+01

maximum values of element variables:
-----------------------------------
max | eps_11 | = 5.124E-05 @ x1 = 1.137E+01, x2 = 1.342E+00
max | eps_22 | = 1.637E-05 @ x1 = 1.137E+01, x2 = 1.342E+00
max | eps_33 | = 1.067E-05 @ x1 = 1.137E+01, x2 = 1.585E-01
max | eps_12 | = 3.171E-05 @ x1 = 1.137E+01, x2 = 5.915E-01
max | sig_11 | = 1.002E+03 @ x1 = 1.137E+01, x2 = 1.585E-01
max | sig_22 | = 1.748E+02 @ x1 = 1.137E+01, x2 = 9.085E-01
max | sig_33 | = 0.000E+00 @ x1 = 6.340E-01, x2 = 1.585E-01
max | sig_12 | = 2.599E+02 @ x1 = 1.137E+01, x2 = 5.915E-01

node : 1 ( x1 = -3.461E-35, x2 = -5.642E-36 )
u_1 = -1.645E-04, u_2 = -1.946E-26
node : 2 ( x1 = 9.028E-36, x2 = 3.750E-01 )
u_1 = -8.161E-05, u_2 = -4.566E-06
node : 3 ( x1 = 5.868E-35, x2 = 7.500E-01 )
u_1 = -2.469E-06, u_2 = -8.382E-06
node : 4 ( x1 = 7.041E-36, x2 = 1.125E+00 )
u_1 = 7.555E-05, u_2 = -9.314E-06
node : 5 ( x1 = -3.259E-35, x2 = 1.500E+00 )
   u_1 = 1.556E-04, u_2 = -9.593E-06

node : 6 ( x1 = 1.500E+00, x2 = -1.652E-36 )
   u_1 = -1.440E-04, u_2 = -3.223E-04

node : 8 ( x1 = 1.500E+00, x2 = 7.500E-01 )
   u_1 = -2.321E-06, u_2 = -3.229E-04

node : 10 ( x1 = 1.500E+00, x2 = 1.500E+00 )
   u_1 = 1.385E-04, u_2 = -3.216E-04

node : 11 ( x1 = 3.000E+00, x2 = 3.616E-36 )
   u_1 = -1.059E-04, u_2 = -5.776E-04

node : 12 ( x1 = 3.000E+00, x2 = 3.750E-01 )
   u_1 = -5.380E-05, u_2 = -5.802E-04

node : 13 ( x1 = 3.000E+00, x2 = 7.500E-01 )
   u_1 = -2.669E-06, u_2 = -5.816E-04

node : 14 ( x1 = 3.000E+00, x2 = 1.125E+00 )
   u_1 = 4.862E-05, u_2 = -5.811E-04

node : 15 ( x1 = 3.000E+00, x2 = 1.500E+00 )
   u_1 = 1.013E-04, u_2 = -5.795E-04

node : 16 ( x1 = 4.500E+00, x2 = 8.784E-36 )
   u_1 = -4.426E-05, u_2 = -7.304E-04

node : 18 ( x1 = 4.500E+00, x2 = 7.500E-01 )
   u_1 = -2.206E-06, u_2 = -7.343E-04

node : 20 ( x1 = 4.500E+00, x2 = 1.500E+00 )
   u_1 = 3.977E-05, u_2 = -7.315E-04

node : 21 ( x1 = 6.000E+00, x2 = 5.143E-38 )
   u_1 = 2.366E-05, u_2 = -7.481E-04

node : 22 ( x1 = 6.000E+00, x2 = 3.750E-01 )
   u_1 = 1.091E-05, u_2 = -7.510E-04

node : 23 ( x1 = 6.000E+00, x2 = 7.500E-01 )
   u_1 = -2.059E-06, u_2 = -7.519E-04
node : 24 ( x1 = 6.000E+00, x2 = 1.125E+00 )
    u_1 = -1.487E-05, u_2 = -7.518E-04

node : 25 ( x1 = 6.000E+00, x2 = 1.500E+00 )
    u_1 = -2.753E-05, u_2 = -7.497E-04

node : 26 ( x1 = 7.500E+00, x2 = 2.383E-36 )
    u_1 = 8.326E-05, u_2 = -6.280E-04

node : 28 ( x1 = 7.500E+00, x2 = 7.500E-01 )
    u_1 = -1.624E-06, u_2 = -6.310E-04

node : 30 ( x1 = 7.500E+00, x2 = 1.500E+00 )
    u_1 = -8.644E-05, u_2 = -6.298E-04

node : 31 ( x1 = 9.000E+00, x2 = 2.961E-36 )
    u_1 = 1.213E-04, u_2 = -4.030E-04

node : 32 ( x1 = 9.000E+00, x2 = 3.750E-01 )
    u_1 = 5.731E-05, u_2 = -4.028E-04

node : 33 ( x1 = 9.000E+00, x2 = 7.500E-01 )
    u_1 = -1.169E-06, u_2 = -4.030E-04

node : 34 ( x1 = 9.000E+00, x2 = 1.125E+00 )
    u_1 = -5.963E-05, u_2 = -4.039E-04

node : 35 ( x1 = 9.000E+00, x2 = 1.500E+00 )
    u_1 = -1.235E-04, u_2 = -4.053E-04

node : 36 ( x1 = 1.050E+01, x2 = -6.019E-36 )
    u_1 = 9.234E-05, u_2 = -1.510E-04

node : 38 ( x1 = 1.050E+01, x2 = 7.500E-01 )
    u_1 = -6.348E-07, u_2 = -1.498E-04

node : 40 ( x1 = 1.050E+01, x2 = 1.500E+00 )
    u_1 = -9.351E-05, u_2 = -1.537E-04

node : 41 ( x1 = 1.200E+01, x2 = -2.069E-36 )
    u_1 = 1.539E-25, u_2 = -1.469E-05

node : 42 ( x1 = 1.200E+01, x2 = 3.750E-01 )
    u_1 = 5.474E-26, u_2 = -6.354E-06
node : 43 ( x1 = 1.200E+01, x2 = 7.500E-01 )
  u_1 = -4.582E-29, u_2 = -7.700E-26

node : 44 ( x1 = 1.200E+01, x2 = 1.125E+00 )
  u_1 = -5.484E-26, u_2 = -7.842E-06

node : 45 ( x1 = 1.200E+01, x2 = 1.500E+00 )
  u_1 = -1.535E-25, u_2 = -1.766E-05

max | u_1 | = 1.645E-04 @ node 1 (-3.461E-35, -5.642E-36)
max | u_2 | = 7.519E-04 @ node 23 (6.000E+00, 7.500E-01)

apes -> end of analysis . . . . . . .

With refinement, the transverse displacement at the beam center (6.000E+00, 7.500E-01) increases from -5.091E-04 to -7.519E-04. This is consistent with the fact that refined model has a greater number of nodal degrees of freedom. As such, the response would be expected to be more “flexible.”
Bibliography
