Lecture #3: Point Source Loading

\[ W \text{ loading rate} = \frac{M}{t} \quad (kg/ha) \]

\[ \frac{dC}{dt} = W - QC - kC/V \]

\[ L^3 \quad M/L^3 \]

Steady State: \( \frac{dC}{dt} = 0 \)

\[ C = \frac{W}{Q + kV} \]

\[ = \frac{W/Q}{1 + kV/Q} \]

\[ = \frac{W/Q}{1 + kt_0} \]

\[ \frac{W}{Q} \text{ is } C \text{ for } k = 0 \]

Special Case:

\[ \frac{W/Q}{1 + kt_0} \quad \text{for } k < 1 \]

\[ \frac{W/Q}{kt_0} \quad \text{for } k > 1 \]

Ex.: \( k = 0.1 \text{ day}^{-1} \) (10% change/day)

\[ t_0 = 1 \text{ day} \]

Residence time too small

Reactor rate too high

NB:

\[ C = \frac{W}{kV} \quad (kt_0 >> 1) \quad = \frac{W/Q}{k V/Q} = \frac{W/Q}{kt_0} \]
Time Variable Behavior (Lec # 4 Chopra)

\[ C(0) = \frac{M}{V} \]

\[ V \frac{dc}{dt} = -Qc - kcVc \]
\[ \frac{dc}{dt} = -\frac{Qc}{V} - kc \]
\[ = -c \left( k + \frac{1}{t_0} \right) \]

\[ C(t) = C(0) e^{-\lambda t} \]
\[ = C(0) e^{-\left( k + \frac{1}{t_0} \right) t} \]
\[ = C(0) e^{-t/\tau} \]
\[ \tau = t_0 \left( k + \frac{1}{t_0} \right) \]

\[ \tau = \frac{1}{k + \frac{1}{t_0}} \]
\[ = \frac{1 + kt_0}{t_0} \]

\[ \frac{C_{55}}{C_{11w}} = \frac{C_{11w}}{1 + kt_0} \]

\[ \tau = \frac{t_0}{1 + kt_0} \left( 1 + kt_0 \right) \]

\[ \tau = +\tau_2 \text{ constant} \]

\[ t = \gamma \nu \]
\[ C(\gamma \nu) = 0.05 C(0) \]

\[ \gamma = \frac{t_0}{1 + kt_0} \left( 1 + kt_0 \right) \]

\[ \tau = \frac{t_0}{1 + kt_0} \left( 1 + kt_0 \right) \]
\[ \tau = \frac{t_0}{1 + kt_0} \quad k t_0 \ll 1 \quad \tau \to t_0 \]

\[ k t_0 \gg 1 \quad \tau \to \frac{1}{k} \]

\[ Q \xrightarrow{t} Q \]

\[ W \xrightarrow{t} W_0 \]

\[ c(t) = \frac{W_0 a}{1 + k t_0} \left(1 - e^{-t/\tau}\right) \]

\[ \frac{dc}{dt} = \frac{W}{V} - \frac{a}{V} c - k c \]

\[ \tau = \frac{t_0}{1 + k t_0} \]

Median

\[ t_0 \approx 100 \text{ days} \]

Treatment Plant

\[ c_{in} \xrightarrow{Q} \xrightarrow{c} \]
\[
\% \text{ Remaining} = \frac{C}{C_{1N}} = \frac{1}{1 + k_{t_0}} \Rightarrow \frac{1}{k_{t_0}} \quad (k_{t_0} >> 1)
\]

How to make \(k_{t_0} >> 1\)

\[k \text{ large to large}
\]

\[
\tau = \frac{t_0}{1 + k_{t_0}} \Rightarrow \frac{1}{k}
\]

Why does the time constant depend only on \(1/k\), not to if \(k_{t_0} >> 1\)?

Principle of Superposition.

For linear differential equations (e.g. involving \(C\) and \(dC/dt\) and \(d^2C/dt^2\) .... but no powers (e.g. \(e^2\)), the solution to a sum of loads is the sum of the solution to each load.
\[ c_1(t) = \frac{W/a}{1 + k t_0} \left( 1 - e^{-t/\tau} \right) \]

\[ c_2(t) = -\frac{W/a}{1 + k t_0} \left( 1 - e^{-\left( t-t_1 \right)/\tau} \right) \]

\[ \tau = \frac{t_0}{1 + k t_0} \]

\[ c_1(t) + c_2(t) = c(t) \]

\[ c(t) = c_1(t) \quad 0 < t < t_1 \]

\[ c(t) = c_1(t) + c_2(t) \quad t > t_1 \]

Example: \[ c = \frac{W/a}{1 + k t_0} \]

Question: What is \( c \) if half \( W \)?

How long? \( c \Rightarrow c/2 \) what is \( \tau \)?
Numerical Solutions of DE's

\[
\frac{dc}{dt} = -kc
\]

\[c(t) = c(0) e^{-kt}\]

**Finite Difference Methods**

Replace \( \frac{dc}{dt} \) with \( \frac{c(t+\Delta t) - c(t)}{\Delta t} \)

**Difference Eq.**

\[
\frac{c(t+\Delta t) - c(t)}{\Delta t} = -kc(t)
\]

Solve for \( c(t+\Delta t) = c(t) - \Delta t \cdot k \cdot c(t) \)

\[
= c(t) (1 - k\Delta t)
\]

**Initial c(0)**

Condition

\[c(\Delta t) = c(0) (1 - k\Delta t) (1 - k(0)\Delta t)\]

\[c(2\Delta t) = c(\Delta t) (1 - k\Delta t) (1 - k(\Delta t)\Delta t)\]

\[\vdots\]

1. Can solve with \( k(t) \) if \( k(t) \) is known
\[ \frac{dc}{dt} = -kc \] 

\[ c(t + \Delta t) = c(t)(1 - k\Delta t) \]

\[ \begin{align*}
\text{What happens if } & \quad k\Delta t > 1 \\
\text{So } & \quad \Delta t \text{ small}
\end{align*} \]

\[ \Delta t \frac{t_o}{t_o} > 1 \]

\[ \Delta t < \frac{t_o}{\phi} \]

\[ \Delta t \ll \tau \]

**General Rule**: \( \Delta t \ll \tau \) (time constant)