Journal of the
SANITARY ENGINEERING DIVISION
Proceedings of the American Society of Civil Engineers

PHOTOSYNTHESIS AND OXYGEN BALANCE IN STREAMS

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INTRODUCTION

The concentration of dissolved oxygen is a significant factor in the evaluation of water pollution and in the management of water quality. In relatively unpolluted waters the dissolved oxygen concentration may fluctuate about the saturation value, a state which results from the dynamic equilibrium among the biological, chemical, and physical components of the environment. There is a great diversity of species, with none one form predominating. Sources and sinks of oxygen are in approximate balance, and the concentration remains close to saturation, with some diurnal variation due to the temperature and photosynthetic activity of green plants. By contrast, in a stream receiving untreated waste waters, the natural balance is upset, bacteria predominant and, among many other effects, a marked depression of the dissolved oxygen results. In many streams, into which waste waters from intermediate or secondary treatment processes are discharged, the dissolved oxygen condition reflects characteristics of both these extremes and it is quite common to observe both bacterial and algal effects. The classic sag in the dissolved oxygen distribution may be observed, superimposed on which is a diurnal fluctuation due to algae growths, stimulated by the nutrients from the waste source or from the end products of earlier reactions. On the one hand, the bacteria produce a typical longitudinal distribution of dissolved oxygen deficit, the components of which are the carbonaceous and nitrogenous sources in the waste water, the latter being more significant in secondary effluents. On the other, the algae cause a typical diurnal fluctuation, which generally increases in amplitude downstream from the source, reaching a peak, and subsequently decreasing to levels characteristic of upstream stretches. It is the specific

Note.—Discussion open until September 1, 1970. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Sanitary Engineering Division, Proceedings of the American Society of Civil Engineers, Vol. 96, No. SA2, April, 1970. Manuscript was submitted for review for possible publication on March 5, 1969.

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purpose of this paper to present a mathematical formulation of the effect of these factors on the dissolved oxygen concentration in rivers, particularly with respect to the photosynthetic effect. The ultimate aim of this study is to understand the stream's response to treated effluents and to develop the equations by which the effects of advanced treatment may be predicted. With background information such as this it would be possible to plan, design, and operate more effectively wastewater treatment plants, and related facilities for the optimal control of water quality in a given drainage basin.

REVIEW OF PREVIOUS WORK

It has long been recognized that the photosynthetic activity of green plants affects the oxygen balance in a stream. In their original analysis of the dissolved balance in a stream, Streeter and Phelps (25) remarked that a "third possible source of available oxygen is the (photosynthetic) activity of certain aquatic plants notably the green and blue-green algae."

A more quantitative early (1933) demonstration of this fact was given by Calvert (2), who collected dissolved oxygen data throughout the day at several stations on the White River in Indiana. A typical diurnal variation of a minimum before dawn to a maximum during midafternoon was observed. Similar observations have been made in lakes (17). The magnitude of this variation, in some cases is negligible, and in others is very significant. The British investigators (12) also have indicated the importance of this factor. A historical review of the subject has been presented (16).

Three different approaches have been generally suggested for a quantitative determination of the photosynthetic production of oxygen in natural bodies of water. The first of these techniques involves the direct measurement of the oxygen production, the oldest example being the light bottle-dark bottle analysis. This approach has the considerable merit of being an attempt at a direct measurement. However, in the dark bottle, the oxygen utilization includes the bacterial as well as algal respiration; furthermore, the lack of light may affect the algae respiration and result in algal mortality (30). In order to maintain the characteristics of the biophysical environment, algal chambers have been constructed (18,20); however, bacterial and mortality effects are retained.

The second method involves the calculation of oxygen production using laboratory determinations of the net photosynthetic rate of oxygen per unit dry weight of aquatic plants (29) or per unit concentration of chlorophyll (23) and the relation of this rate to sunlight intensity. This method requires a large number of relationships and assumptions. However, it has the important advantage of being a determination which is independent of the observed dissolved oxygen profiles and can in principle be used as an input into a mathematical model of dissolved oxygen behavior.

The third technique is based on the principle of a mass balance of oxygen. If all the known nonalgae components which are sources and sinks of oxygen are subtracted from the observed diurnal oxygen curve or curves, then the remaining source must be of algal origin. This technique first formulated by Odum (22) to include respiration has been extended to include BOD as a sink of oxygen (18).

3 Numerals in parentheses refer to corresponding items in the Appendix III—References.
In this paper, the functional forms of the photosynthetic oxygen source and respiration oxygen sink are hypothesized a priori with the proportionality constant of each term as unknowns. Instead of back calculating the functional forms of photosynthetic production and respiration, a general form for these functions is hypothesized and only the proportionality constants are evaluated from the data. The procedure permits utilization of all the available data; in particular, the organic solution, both carbonaceous and nitrogenous, which may be present, and the qualitative or quantitative information about the algal distributions in space. This analysis is, therefore, an extension of the technique based on the continuity equation for dissolved oxygen which includes the diurnal time-variable effect of photosynthetic oxygen production. Previous reports have incorporated the average daily photosynthetic oxygen and respiration in a spatial analysis (9,10). The following analysis considers the temporal as well as spatial distributions. The significance and evaluation of the remaining sources and sinks of dissolved oxygen have been presented (20).

THEORY

The most general form of equation which describes the longitudinal distribution of dissolved oxygen or any nonconservative or conservative substance is developed by a mass balance employing the continuity equation

\[ \frac{\partial c}{\partial t} + \frac{1}{A(x,t)} \frac{\partial}{\partial x} (Q(x,t)c) + S(c,x,t) = 0 \]  

in which \( c \) = concentration; \( A \) = cross-sectional area; \( Q \) = freshwater flow; and \( S \) = sources and sinks. Eq. 1 applies to a freshwater stream in which the dispersive component of the flux is small by contrast to the advective component. The concentration of the substance is assumed to be uniform in the lateral and vertical planes. The flow and the cross-sectional area may have spatial and temporal variation. Similarly, sources and sinks may be functions of time and space, and, in addition, functions of its own concentration or the concentration of other substances. Eq. 1 is a fundamental expression for the concentration of any substance, conservative or nonconservative, in a one-dimensional stream, in which the advection is the only significant component of the flux. The specific form of the equation for a given stream is determined by the hydraulic, hydrological, and geomorphological characteristics of the drainage basin and river channel, on the one hand, and the various physical, chemical, and biological characteristics of the aquatic environment and wastewater discharges on the other. For the rivers and conditions considered in this paper, Eq. 1 takes on the following form for the concentration of dissolved oxygen:

\[ \frac{\partial c}{\partial t} = - \frac{Q}{A} \frac{\partial c}{\partial x} + K_d(c_s - c) + K_dL(x) - K_BN(x) + P(x,t) - R(x) - S(x) \]

in which \( c \) = concentration of dissolved oxygen; \( c_s \) = saturation value of dissolved oxygen; \( K_d \) = reaeration coefficient; \( K_d \) = coefficient of carbonaceous oxidation; \( L(x) = L_0 e^{-K_d\sqrt{U}} \) = distribution of carbonaceous BOD; \( U = Q/A; \) \( K_B \) = coefficient of nitrogenous oxidation; \( N(x) = N_0 e^{-K_B\sqrt{U}} \) = distribution of nitrogenous BOD; \( U = Q/A; \) \( P(x,t) \) = algal photosynthetic oxygen source; \( R(x) \) = algal respiration sink; and \( S(x) \) = benthic bacterial respiration sink.

FIG. 1.—OBSERVED AND CALCULATED TEMPORAL DISTRIBUTIONS OF DISSOLVED OXYGEN

planktonic and benthic environments. The bacterial respiration is composed of carbonaceous and nitrogenous components. The bacterial respiration of benthic community is assigned as a separate sink. The algae in both floating and stationary form are included as a single source. For the sake of mathe-
mational simplicity the spatial distributions of both the photosynthetic source and the respiration sink are assumed to begin abruptly at $x = 0$ and remain constant for $x > 0$. The effect of any upstream algal activity, i.e., for $x < 0$, is taken into account in the boundary condition at $x = 0$. The temporal form of the respiration sink is assumed to be a constant $R$. This is the traditional assumption and it appears to be justifiable within the accuracies of the data so far available. The temporal variation of the photosynthetic oxygen source is a result of the solar radiation incident on the algae present. The variation of oxygen production with increasing solar radiation and the attenuation of solar radiation with increasing depth have been investigated (24). With increasing solar radiation the rate of oxygen production increases to a saturation value and then decreases. For a bright day, at the top layer of water, the oxygen production may be less at midday than during the morning and afternoon. However, solar radiation decreases with depth so that in the middle of the euphotic zone the oxygen may be more or less proportional to the sunlight intensity. If the average over depth is taken, then the shape of the cross-sectional average photosynthetic oxygen production rate resembles the shape of the incident solar radiation (23). For the sake of simplicity the temporal form of the photosynthetic oxygen source is assumed to be representable by a half cycle sine wave. If more complete information is available this assumption can be refined. Let $P(t)$ represent the rate of photosynthetic oxygen production as a function of time, thus $P(t)$ is given by

$$P(t) = P_m \sin \left( \frac{\pi}{p} (t - t_s) \right) \text{ when } t_s = t = t_g + p \quad \text{or} \quad \sum_{n=1}^{\infty} b_n \cos \left[ 2\pi n \left( t - t_g - \frac{p}{2} \right) \right]$$

(3)

in which $P_m$ = the maximum rate of photosynthetic oxygen production, in milligrams per liter-day; $t_g$ = the time at which the source begins, in days; and $p$ = the fraction of the day over which the source is active, in days. This function is assumed to repeat periodically every day. An example is shown in Fig. 1.

Eq. 3 specifies $P(t)$ for 1 day only; its periodic extension can be expressed as a Fourier series (see Appendix I):

$$P(t) = P_m \sum_{n=1}^{\infty} b_n \cos \left[ 2\pi n \left( t - t_g - \frac{p}{2} \right) \right]$$

(4a)

in which

$$b_n = \cos \left( \frac{\pi n}{p} \right)$$

(4b)

Using this expression for the photosynthetic oxygen source and expressing Eq. 2 in terms of the dissolved oxygen deficit, $D(x,t) = c_b - c(x,t)$, results in

$$\frac{\partial D}{\partial t} + \frac{Q}{A} \frac{\partial D}{\partial x} = -K_d D + K_d L(x) + K_n N(x) + S - P(t) + R$$

(5)

in which $S$ is assumed to be constant.

For an initial condition $D(x,0) = 0$, $x > 0$, and an arbitrary boundary condition $D(x,0)$, Eq. 5 has as its solution

$$D(x,t) = D_0 \left( \frac{1}{2} \right) e^{-K_d U(x/t)} + \frac{K_a L_0}{K_a} \left[ e^{-K_d s(x/t)} - e^{-K_d (x/U)} \right] + \frac{K_n}{K_a} \left[ e^{-K_d s(x/U)} - e^{-K_d (x/U)} \right] + \frac{S}{K_a} \left[ 1 - e^{-K_d s(x/U)} \right] + R \left[ 1 - e^{-K_d s(x/U)} \right]$$

$$- \frac{F_m}{\pi K_a} \left[ 1 - e^{-K_d s(x/U)} \right]$$

$$= \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{K_a^2 + (2\pi n)^2}} \cos \left[ 2\pi n \left( t - t_g - \frac{p}{2} \right) \right] \tan^{-1} \left( \frac{2\pi n}{K_a} \right)$$

(6)

in which $b_n$ is given by Eq. 40. The time varying portion of this solution is developed in Appendix B. It has been assumed in this solution that the period photosynthetic source has been active for a long time, i.e., a periodic steady state has been achieved.

The second, third, and fourth lines of Eq. 6 are the well known deficit solutions due to BOD and benchic sinks of DO. Line one of Eq. 6 is the solution for a time-varying initial deficit of DO. The portion of the dissolved oxygen deficit solution due to the photosynthetic source and respiration sink is made up of three parts. The average daily photosynthetic oxygen production (line 6 of Eq. 6) and the respiration (line 5 of Eq. 6) portion of the solution are constant in time and build up to their equilibrium values of $P_m/K_a$ and $R/K_a$, respectively, as $x/U$ increases. Line 7 of Eq. 6 is independent of position and represents the solution after the effect of the initial abrupt beginning of the photosynthetic source at $x = 0$ has decayed to zero. The portion of the solution which is due to the abrupt beginning, (line 8 of Eq. 6) is multiplied by an exponential which goes to zero as $x/U$ increases. For large $K_a$, as in the case in the streams to be analyzed herein, the diurnal fluctuation of the solution due to the photosynthetic source has zero amplitude at $x = 0$ and increases to the steady state amplitude given by line 7 of Eq. 6.

The following effects have not been included in this solution; the time variation of the flow (9), the time variation of the temperature and wastewater discharges, and the effect of dispersion. In the streams to which this solution is applied it is assumed that the primary cause of the diurnal variation of the dissolved oxygen is the algal oxygen production.

**ANALYSIS OF DATA**

Five rivers have been selected for analysis, the Grand River (16), the Clinton River (15) and the Flint River (7) in Michigan; the Truckee River (19).
in Nevada, and the Ivel River (13) in Great Britain. The selection is based on the availability of historical data which is sufficient for the proposed analysis, with emphasis being on the diurnal behavior of dissolved oxygen and the effect of the nitrogenous BOD component. The data available for analysis was observed during periods when a reasonable approximation of a steady-state flow and temperature prevailed.

![Cross sectional area, depth, observed and calculated spatial dissolved oxygen profile](image)

The average cross-sectional area for each river except the Flint and the Ivel is obtained from the reported time of passage studies and average river flows. For the Flint River, detailed cross-sectional information, Fig. 2, is available (1), for the Ivel the average velocity is estimated from the time of travel reported. The average depths are reported in all cases except the

**TABLE 1.—HYDRAULIC AND PHYSICAL CHARACTERISTICS**

| River | Segment number | Segment location, in miles | Segment length, in miles | Cross-sectional area, A | Depth, D, in feet | Temperature, T, in degrees Celsius | Discharge, Q, in cubic feet per second | Residence, K, per day | S|N|B|O
<table>
<thead>
<tr>
<th></th>
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<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Grand</td>
<td>1 &amp; 2</td>
<td>0.0</td>
<td>12.0</td>
<td>320.0</td>
<td>1.50</td>
<td>28.0</td>
<td>285.0</td>
<td>5.5</td>
<td>1.00</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>Clinton</td>
<td>1</td>
<td>0.0</td>
<td>5.5</td>
<td>44.0</td>
<td>1.16</td>
<td>21.0</td>
<td>195.0</td>
<td>0.5</td>
<td>2.5</td>
<td>-</td>
<td></td>
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<tr>
<td>Truckee</td>
<td>2</td>
<td>5.5</td>
<td>5.5</td>
<td>150.0</td>
<td>1.50</td>
<td>27.8</td>
<td>180.0</td>
<td>0.5</td>
<td>2.5</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Ivel</td>
<td>1</td>
<td>0.0</td>
<td>10.1</td>
<td>150.0</td>
<td>1.67</td>
<td>27.8</td>
<td>172.0</td>
<td>1.3</td>
<td>2.4</td>
<td>-</td>
<td></td>
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<tr>
<td>Flint</td>
<td>2</td>
<td>16.2</td>
<td>2.0</td>
<td>400.0</td>
<td>1.20</td>
<td>28.0</td>
<td>284.0</td>
<td>0.5</td>
<td>2.5</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

* Initial BOD estimated from river data.
* Initial N=O calculated using W/Q.
* Initial N=O estimated from river data.
* Initial N=O calculated using W/Q.
* Initial N=O estimated from river data.

**TABLE 2.—FOURIER COEFFICIENT FOR INITIAL CONDITION OF DISSOLVED OXYGEN**

<table>
<thead>
<tr>
<th>River</th>
<th>A&lt;sub&gt;0&lt;/sub&gt;, in milligrams per liter</th>
<th>A&lt;sub&gt;1&lt;/sub&gt;, in milligrams per liter</th>
<th>A&lt;sub&gt;2&lt;/sub&gt;, in milligrams per liter</th>
<th>θ&lt;sub&gt;0&lt;/sub&gt;, in days</th>
<th>θ&lt;sub&gt;1&lt;/sub&gt;, in days</th>
<th>θ&lt;sub&gt;2&lt;/sub&gt;, in days</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grand</td>
<td>3.5</td>
<td>-</td>
<td>-</td>
<td>-0.110</td>
<td>0.255</td>
<td>-0.204</td>
</tr>
<tr>
<td>Clinton</td>
<td>4.46</td>
<td>0.823</td>
<td>-0.010</td>
<td>-0.010</td>
<td>-0.010</td>
<td>-0.010</td>
</tr>
<tr>
<td>Truckee</td>
<td>7.97</td>
<td>1.04</td>
<td>1.12</td>
<td>1.12</td>
<td>1.12</td>
<td>1.12</td>
</tr>
<tr>
<td>Ivel</td>
<td>10.43</td>
<td>1.27</td>
<td>-0.0001</td>
<td>0.157</td>
<td>0.157</td>
<td>0.157</td>
</tr>
<tr>
<td>Flint</td>
<td>6.66</td>
<td>1.12</td>
<td>-0.122</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
</tbody>
</table>

\[ a(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin \left( 2\pi k \left( t - \theta_k \right) \right) \]

these data, the reaction coefficient is calculated from (21)

\[ K_a = \frac{(D_Q L)^{1/3}}{B^{1/3}} \]
The reported average temperatures are then used to correct this value to the stream condition during the survey and establish the saturation value of dissolved oxygen. The coefficient of 1.025 was used to make the temperature adjustment for the resaturation coefficient ($S$).

The procedure followed for the analysis of the carbonaceous and nitrogenous BOD data relies on the observed river values, although known efficient concentrations and flows from treatment plant in the reaches analyzed are used to establish loads. The carbonaceous and nitrogenous reaction coefficients, $K_{C}$ and $K_{N}$, are determined from a plot of the logarithm of the particulate component of the BOD versus distance from the source. The slope of the line, multiplied by the velocity, yields the coefficient. The nitrogenous and carbonaceous components of the BOD data are obtained in a variety of ways: nitrification suppression using methylene blue for the Grand; estimated ultimate first stage BOD, organic nitrogen and ammonia measurements for the Truckee; 5-day BOD; and some ammonia measurements for the Flint. There

<table>
<thead>
<tr>
<th>River</th>
<th>Segment code</th>
<th>$P_{m}$ (milligrams per liter-day)</th>
<th>$P_{w}$ (milligrams per liter-day)</th>
<th>$R$ (milligrams per liter-day)</th>
<th>$P_{w}/R$</th>
<th>$t_{s}$ (per hour)</th>
<th>$P_{r}$ (per hour)</th>
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<td>7.6</td>
<td>22.0</td>
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<td>0.90</td>
<td>12</td>
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<td></td>
<td>2</td>
<td>50.0</td>
<td>17.3</td>
<td>16.0</td>
<td>1.1</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>60.0</td>
<td>22.5</td>
<td>16.0</td>
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<td></td>
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<tr>
<td>Clinton</td>
<td>1</td>
<td>27.0</td>
<td>8.6</td>
<td>20.0</td>
<td>0.43</td>
<td>0.60</td>
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<td></td>
<td>2</td>
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<td>15.0</td>
<td>18.0</td>
<td>0.98</td>
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<tr>
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<td>11.2</td>
<td>7.0</td>
<td>1.6</td>
<td>0.50</td>
<td>14</td>
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<td></td>
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<td>15.0</td>
<td>3.0</td>
<td>0.70</td>
<td>14</td>
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<tr>
<td>Fleet</td>
<td>1</td>
<td>35.0</td>
<td>11.1</td>
<td>15.0</td>
<td>0.91</td>
<td>0.60</td>
<td>12</td>
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<tr>
<td></td>
<td>2</td>
<td>50.0</td>
<td>17.5</td>
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<td>0.92</td>
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<td></td>
<td>3</td>
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<td>19.1</td>
<td>17.0</td>
<td>1.11</td>
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<td></td>
<td>4</td>
<td>70.0</td>
<td>22.3</td>
<td>20.0</td>
<td>1.11</td>
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</tbody>
</table>

* See Table 1 for hydraulic and physical data of segments.

is no reported nitrogen data for the Clinton, and the Ivel is relatively unpolluted, having no significant concentration of BOD. The concentrations, $L$, and $N_{0}$ were taken from the direct measurements and checked whenever possible, by a mass balance upstream and downstream of the waste discharge.

The BOD analysis establishes the sinks of oxygen due to the carbonaceous and nitrogenous BOD. There remains the algal source and sink of oxygen. The procedure adopted to obtain $P_{m}$ and $R$ is largely one of adaption and error based primarily on knowledge of the magnitude of the diurnal fluctuation at the various stations from dawn to sunset and guided by whatever qualitative information is available about the algal populations. The two times $t_{s}$ and $P_{r}$ are established using the time of sunrise and sunset. It has been found that the best fits are obtained if $t_{s}$ is slightly later than sunrise and if $P_{r}$ is slightly later than the total time between sunrise and sunset. This is probably due to the fact that the assumed shape of

the photosynthetic oxygen source (Eq. 4) rises and sets more quickly than apparently is the case.

The procedure is as follows: the diurnal dissolved oxygen data at the first station establishes the boundary condition. This condition, the carbonaceous and nitrogenous BOD sinks, and assumed values of $P_{m}$ and $R$ are used for the first year segment of river. A uniform segment ends at either a load which alters the BOD concentrations and possibly the structure of the algal populations, or a discontinuity in area or depth, or a discontinuity in $P_{m}$ and $R$ reflecting a change in algal productivity. At such a point the values of the parameters change and numerical calculations are initiated again. The boundary condition used is the diurnal distribution calculated for the end of the previous segment. In effect, this procedure makes possible the solution of more general stream configurations which are limited only to piecewise constant spatial parameters. Using this technique the values of $P_{m}$ and $R$ which best fit the data are chosen. The choice of $P_{m}$ is guided by the fact that the magnitude of the diurnal fluctuation at a station is directly proportional to $P_{m}$ so the change in $P_{m}$ which is required to fit the data can be calculated after one of the other sources and sinks. Since the respiration is assumed to be a constant in time, its effect is only to lower the average daily value of the DO. Therefore the appropriate value can be obtained from a spatial plot of the time averaged data. Using this procedure, then, it is possible to choose $P_{m}$ and $R$ more independently since the $P_{m}$ is chosen using the magnitude of the diurnal fluctuations as a guide and the $R$ is chosen using the daily average values. Therefore under the assumption that respiration is a constant in time it is possible to separate these two algal effects. It should be recognized, however, that the respiration is, in effect, all respiration not accounted for by other sinks of DO, and that to that extent is not necessarily only algal respiration. For example, it may be benthic respiration which has not taken explicitly into account. On the other hand, the final choice of $R$ does reflect all that is known about the river oxygen balance, e.g., benthic respiration can be independently assigned, and to that extent, it can be reliably estimated.

Tabulations of hydraulic characteristics, the parameters, and the boundary conditions employed in this analysis are presented as follows: Table 1 includes the location of beginning of the segments; their lengths $X_{i}$; cross-sectional areas $A_{i}$; discharges $Q_{i}$; reaeration coefficients $K_{a}$; and carbonaceous and nitrogenous reaction coefficients $K_{C}$ and $K_{N}$. The time variable initial conditions of dissolved oxygen are presented in Table 2. Table 3 lists the parameters defining the algae effects: the time, $t_{s}$, at which the photosynthetic oxygen source begins; the period, $p$, over which the source is nonzero; the maximum rate, $P_{m}$, of photosynthetic oxygen production; the daily average rate of photosynthetic oxygen production, $P_{d} = 2pP_{m}/p$; the rate of algal respiration, $R_{i}$; and the ratio of $P_{m}$ to $R$. The Grand River is located in the southwestern part of Michigan, flows westery to Lake Michigan and has a drainage area of 5,620 sq miles. The City of Lansing is located approximately 15 miles upstream from the mouth of the river. The analysis of a survey conducted in August, 1960, extends from the discharge of the waste treatment plant to a point about 25 miles downstream. The treatment plant is a biological process of the activated sludge type. The possibility of nitrification as an important factor in the
Oxygen balance was recognized at the time of the survey and nitrification suppressed BOD's by methylene blue were also obtained. No benthic respiration is assumed since the river velocities prior to the survey were large enough to scour the bottom. Insufficient information is available upstream of the Lansing waste discharge to include this region in the analysis. The

![Graph showing dissolved oxygen and nitrogenous oxygen over time](image)

**FIG. 3.—OBSERVED AND CALCULATED SPATIAL PROFILES OF BOD AND DISSOLVED OXYGEN**

analysis is begun at the Lansing treatment plant using river conditions extrapolated from downstream values as the initial conditions. No diurnal DO fluctuation is assumed for the boundary condition. The reach up to the Grand Ledge Dam, Stations 1, 8, 9, and 10, is assumed to be spatially uniform. Downstream of the dam, Stations 11, 12, 13, and 14, profuse attached aquatic plants alter the composition of the algal community and it becomes much more productive.

Accordingly at the dam a new segment is begun with different \( P_{on} \) and \( R \). A comparison of the calculated profiles and observed data are shown in Figs. 3 and 4.

In particular it is clear that the nitrogenous BOD is an important factor in the oxygen balance in streams. For the second segment analyzed two curves

![Graph showing temporal distributions of dissolved oxygen](image)

**FIG. 4.—OBSERVED AND CALCULATED TEMPORAL DISTRIBUTIONS OF DISSOLVED OXYGEN**

are shown corresponding to \( P_{on} = 50 \) and \( P_{on} = 65 \) mg per liter-day. These two curves are intended to describe the data taken on the two successive days, August 29th and August 30th. The solar radiation data available (27), Fig. 3, indicates that the second day was indeed brighter than the first; therefore, a
larger production of oxygen is expected. This effect is not clearly discernible in the data upstream of the dam. For the first segment the ratio of average daily photosynthetic production of oxygen to the average daily respiration \( P_{\text{air}} / R \) equals 0.35, indicating that the planktonic community is a net exporter of the dissolved oxygen resources of the stream. Downstream from the dam the ratio equals 1.1 and 1.4, respectively, which implies the converse. In this stretch of the stream the algal community includes benthic algae and larger aquatic plants as well as any planktonic algae which may be present.

Clinton River.—The Clinton River, located in the southeast corner of Michigan in the Lower Peninsula, discharges into Lake St. Clair. The basin has a drainage area of approximately 740 sq miles. The reach considered in this analysis of a survey conducted during August 23 to August 25, 1960 starts at mile point 44.87, Station 5, below the Pontiac Waste Treatment Plant effluent at mile point 46.39, and ends at mile point 34.98, Station 8. A boundary condition is established at Station 2 by fitting a two-term Fourier series to the observed dissolved oxygen data. The reach including Stations 3, 4, 5 and 6 is assumed to be spatially uniform. The aquatic vegetation in this stretch seems to be mainly attached bottom plants. Between Station 6 and Station 7 the algal population appears to increase in productivity and the survey report cites "numerous weed beds" in this reach. To account for this spatial change a new segment is established 0.5 miles below Station 6 and includes Stations 7 and 8. In this segment the productivity increases markedly while the respiration decreases slightly. The ratio of \( P_{\text{air}} / R \) changes from 0.43 to 0.84. This pattern is similar to that encountered in the analysis of the Grand River. The resulting theoretical curves are compared to the data in Fig. 4.

Truckee River.—The Truckee River is located near the western border of Nevada and flows in a northeasterly direction from Lake Tahoe to Pyramid Lake, Nevada, a distance of 106 miles. The reach considered in this analysis begins at mile point 25.8 (river miles below the Reno USGS gage) designated Station 1 and ends at Station 7, mile point 21.0. The survey was conducted in the period from July 10 to July 25, 1962. Within the reach there are three principle sources of waste water: the Reno Waste Treatment Plant effluent at mile point 0.1, the effluent from the Sparks Waste Treatment Plant which enters at mile point 6.0 and Steamboat Creek entering at mile point 5.7 whose flow is partially irrigation return waters and agriculture drainage. The latter two sources will be treated as one combined source at mile point 5.9. For this analysis, therefore, the reach will be divided into three sections: Station 1 to the Reno Waste Treatment Plant; Reno to Sparks-Steinboat Creek, and Sparks-Steinboat Creek to Station 7.

The analysis of the carbonaceous BOD follows that given in the report of the survey. Five-day BOD's, corrected for the onset of nitrification, are used to obtain ultimate carbonaceous BOD. The available nitrogen data comprises measurements of organic nitrogen, ammonia nitrogen, nitrate nitrogen and nitrite nitrogen. No appreciable ammonia nitrogen is found in section 1. At the beginning of section 2 the abrupt increase of ammonia nitrogen can be attributed to the Reno effluent. There follows a substantial decrease of both ammonia nitrogen and total nitrogen in section 2. For section 3 ammonia is seen to decrease further while total nitrogen is approximately constant. The nitrate values increase correspondingly. The ammonia nitrogen values can be converted to their nitrogenous BOD equivalents using a stoichiometric balance (NBOD = 4.6 × NH₃ - N). In order to account for the decrease in total nitrogen in Section 2 it is hypothesized that the benthic algae in this section are metabolizing that fraction of the ammonia necessary to account for the decrease in total nitrogen. This is taken into account by introducing a benthic sink of NBOD in Section 2 only. The remaining NBOD is assumed to oxidize to nitrate following a first order reaction.

![Figure 5: Observed and Calculated Spatial Distributions of Ammonia and Total Nitrogen, Carbonaceous BOD, and Dissolved Oxygen](image)

The algae population in the three sections are described in the survey report: Station 2: "... abnormally profuse growth of a filamentous alga (Cladophora)"; Station 3: "... lack of Cladophora... in its place the tolerant blue-green Oscillatoria and the green filamentous alga Stigeoclonium were quite evident"; Stations 5, 6, and 7: "The tolerant algal forms found at Station 3 were not observed at Stations 5, 6, and 7, however growths of Cladophora were
present. Sparse growths of a rooted aquatic pondweed (Potamogeton) were present at Stations 5 and 6, but not at Station 7. Photoplankton samples at three stations (1A, 3, and 5) within the reach were analyzed; diatoms were the dominant form and the number per milliliter were, respectively, 1,310, 7,680, and 6,380. Therefore, the division of the reach into three sections also coincides with the changes in algal populations.

An interesting phenomena is observed to occur between Station 4 upstream of the combined load at mile point 5.9 and Station 5 the next downstream at mile point 8.7. The diurnal fluctuation of DO at Station 4 is 7 mg per l while at Station 5 it is 2 mg per l. This decrease in diurnal fluctuation cannot be explained as a result of a decrease in $P_m$ because the effects of the upstream fluctuation at Station 4 would still be quite significant at Station 5. The decrease in diurnal fluctuation is due to the low concentration of DO of the incoming wastewater which not only lowers the average value of DO but also decreases the amplitude of the diurnal fluctuation. The value of $P_m$ in section 3 decreases as can be seen from the distribution of DO at Stations 6 and 7. The theoretical curves are compared to the data in Figs. 5 and 6.

In the O'Connell and Thomas' (18) analysis of the reach between Stations 2 and 4, the values of maximum photosynthetic production $P_m$ and average daily respiration $R$, as shown in their Fig. 5, are $P_m = 50$ mg per l-day and $R = 17$
mg per 1-day for Station 26, which are comparable to the values arrived at in the above analysis. However, as shown in their Fig. 6, they also obtain a spatially varying photosynthetic production and respiration. Unfortunately detailed dissolved oxygen data between Stations 2, 3 and 4 is not available in the survey report so further analysis is not possible. There are good physical reasons to suppose that the algal photosynthetic source and sink do not change abruptly as has been assumed herein, however, for the data currently available this simplification is warranted.

Ivel River.—The Ivel River, located in Great Britain, is a shallow chalk stream. Diurnal dissolved oxygen data are available at two stations; each station was sampled every 15 min for 24 hr during May, 1959. Following the same procedure as previously outlined, the upstream station diurnal dissolved oxygen data is approximated using a two-term Fourier series and the data at the downstream station is compared to the theoretical prediction. The results are shown in Fig. 7; the parameters are listed in Tables 1 and 2. In order to fit the data, the period of sunlight is extended 1 hr more than is justified from the solar radiation data. Because the solar radiation measurements were taken 16 miles from the river site there is a possibility that this data is not representative. A comparison can be obtained between the photosynthetic production and respiration rates estimated by Edwards and Owens (13), and those indicated in Table 3. Unfortunately the average depth of the Ivel is only an estimate based on the reported exchange coefficient and formula for the reseruration coefficient (Eq. 7). However, using this value of $H = 0.45$ ms, the Edwards’ and Owens’ estimates in concentration units are $P_m = 58$ mg per 1-day and $R = 17$ mg per 1-day which are comparable to the estimates arrived at by this analysis.

Flint River.—The Flint River, located in the eastern-central portion of Michigan, has a drainage area of 1,350 sq mile. The reach analyzed in this paper extends from Station 6, Stevenson St, Flint to Station 15, Dodge Rd of the July 26–29, 1959 Survey of the Michigan Water Resources Commission. River cross sections at 500-ft intervals are available and the cross-sectional area and depth are presented in Fig. 2. The variation of the cross-sectional area and depth as a function of distance downstream necessitates the division of the reach under analysis into five uniform segments which approximately account for the observed variations. The two major point sources of BOD and DO deficit are the Flint and the Flushing Sewage Treatment Plants, both of which are trickling filters. The carbonaceous and nitrogenous BOD concentration of the effluent of the Flint STP is reported; the corresponding information for the Flushing STP is estimated from the available river BOD and nitrogen data. The existence of a benthic deposit is reported below the Flint STP and its effect on the oxygen balance is estimated from the river DO data. Although the carbonaceous BOD information is adequately representative, the nitrogenous BOD information is somewhat limited, comprising four ammonia concentrations taken a week before the actual survey.

At the beginning of section 1, the initial conditions are established from the reported river data (Table 2). A relatively high rate of nitrogenous BOD oxidation in this section is confirmed by the available river ammonia data. The Flint load marks the beginning of section 2. As usual materials balance between the reported loads of CJOD, NBOD, and an assumed dissolved oxygen concentration of zero in the effluent with calculated distributions establishes the initial conditions. In this short segment, a benthic respiration load due to a reported sludge deposit is included. Because no independent measurement of the oxygen uptake rate of the sludge deposit is available it is difficult to distinguish its effect on the river dissolved oxygen from the algal respiration. Hence the assumed benthic and algal respirations are arbitrarily assigned although the magnitude of their sum is that required by the observed oxygen data. The diurnal distribution of dissolved oxygen at Station 10 is a result of both a decrease of algal productivity due to water quality conditions which are not conducive to algal growths and the deficit contributed by the Flint treatment plant. This phenomenon has also been observed in the Truckee.

The dissolved oxygen data in section 3 reflects the increased photosynthetic production of oxygen. The authors report that this section of the river is
nearly choked with submerged grass-like aquatic vegetation probably Vallisneria or Sagittaria." As a consequence the afternoon dissolved oxygen distribution reaches nearly a saturated level at the end of the section. However the morning distribution shows almost zero DO throughout the stretch. With these low DO concentrations prevailing during part of the day it is reasonable to assume that no appreciable nitrification is occurring in this section. Although not explicitly confirmed by the river ammonia data, this assumption seems justifiable in light of the concentrations of both ammonia and dissolved oxygen in the section.

It is interesting to note that there are significant departures from the theoretical diurnal dissolved oxygen distribution and the observed distribution at Station 12 (see Fig. 8). Here the DO is substantially zero during the late evening and early morning hours. This low dissolved oxygen appears to be affecting the bacterial or algal activity, or both, as indicated by the shape of the diurnal dissolved oxygen curve.

The dissolved oxygen data at Stations 13 and 14 in section 4 after the Flashing STP show a recovery and also indicate that a phytoplankton activity is significant. The amount of rooted aquatic vegetation is reported as sparse; yet the diurnal dissolved oxygen data indicate that the average photosynthetic oxygen production is $P_{av} = 19$. In section 5, the average daily oxygen photosynthetic production increases to $P_{av} = 22$, in accordance with the observed increase in the amount of grass-like aquatic plants from sparse to moderate. The spatial distribution of dissolved oxygen at the two times reflecting the extreme conditions are presented in Fig. 2. The theoretical diurnal curves are compared to the data in Fig. 8.

CONCLUSIONS

The computations based on the proposed formulation agree reasonably well with observations in various fresh water streams. Both carbonaceous and nitrogenous components of bacterial activity, atmospheric reaeration, benthic deposits and the photosynthetic and respiratory contribution of the algae and other aquatic plants have been considered. Although primarily directed to an analysis of the dissolved oxygen condition, the analysis may be used for productivity and respiration studies of aquatic communities. The photosynthetic effect can be the dominant factor in the diurnal variation of dissolved oxygen as shown by the rivers investigated in this paper. These effects are primarily due to algae, although larger rooted plants may be important in some cases. The growth of the aquatic vegetation is stimulated by nutrients discharged to the streams from the various sources within the drainage basin, particularly the influent waters from treatment plants.

Correlations of computed profiles and observed data in both the spatial and temporal domains are favorable. Therefore, more confidence can be placed in the ability to predict the effects of waste water discharges on receiving waters. Particularly important is the effect of the demand of the nitrogenous component on the oxygen resources. It is significant to date that all of the streams analyzed in this paper receive effluents from biological treatment plants. The importance of nitrification has been demonstrated indicating the need to take this factor into account with respect to secondary treatment. It can be generally expected that this pattern of increased algal activity and nitrification will occur when secondary treatment facilities are installed. The method of analysis presented provides a further technique for water pollution analysis and should be useful in comprehensive planning of water pollution control facilities.

Further work should be directed to analysis of stream conditions before and after treatment facilities are constructed and of time variable phenomena such as fresh water flow, temperature, and waste water discharge.

ACKNOWLEDGMENT

This work was supported by a research grant from the Federal Water Pollution Control Administration.

APPENDIX I.—FOURIER ANALYSIS OF PHOTOSYNTHETIC OXYGEN SOURCE

It has been assumed that the variation of photosynthetic oxygen production through the day is given by $E_{max}(t)$ where $g(t)$ is defined as

$$g(t) = \sin \left( \frac{\pi t}{p} \right) 0 \leq t \leq p \tag{8}$$

$$= 0 \quad p \leq t \leq 1$$

For simplicity let the time $t = 0$ be the start of the oxygen production, usually taken as slightly after sunrise, and let $p$ be the fraction of the day during which oxygen is produced. This function is assumed to repeat daily, and the extended function will also be denoted by $g(t)$. It has period $T = 1$ day. Because $g(t)$ is a periodic function it can be expressed as a Fourier series. Let

$$f(t) = \sum_{n=1}^{\infty} b_n \cos \left( n \omega_o t \right) \tag{9}$$

in which $f(t)$ is now an even periodic function of time also with period $T = 1$ day. The introduction of $f(t)$ is made to simplify the Fourier series representation of $g(t)$. Because $f(t)$ is an even function of time, i.e., $f(t) = f(-t)$, its Fourier series expansion can be given in terms of only cosines (25), i.e.

$$f(t) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos \left( n \omega_o t \right) \tag{10}$$

in which $b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \left( n \omega_o t \right) dt \tag{11}$

and $\omega_o = \frac{2\pi}{T}$ rad per day

By definition of $g(t)$, Eq. 9, and the definition of $f(t)$, $f(t) = \cos \left( \frac{\pi t}{p} \right)$ when $-p/2 \leq t \leq p/2$ and zero outside this interval. Therefore, Eq. 11 becomes
\[ b_h = 2 \int_{-p/2}^{p/2} \cos \left( \frac{\pi t}{p} \right) \cos \left( k \omega_d t \right) dt \] \hspace{1cm} (13)

This integral is readily evaluated (11) and the result, after simplification using the formula for the sine of the sum and difference of two angles, is

\[ b_h = \cos \left( kn \pi \right) \frac{2 \pi}{\left( \frac{p}{n} \right)^2 - (2n)^2} \] \hspace{1cm} (14)

For the special case where \( p = 1/2 \), \( b_h \) is indeterminate. A similar derivation shows that for \( p = 1/2 \), \( b_h = 1/2 \).

Therefore, the Fourier series expansion of \( P_{mg}(t) \), the photosynthetic oxygen source, is given by

\[ P_{mg}(t) = \frac{b_p}{2} + \sum_{n=1}^{\infty} b_n \cos \left( \frac{2n \pi \left( t - \frac{p}{2} \right)}{p} \right) \] \hspace{1cm} (15)

in which \( b_n \) is given by Eq. 14.

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**APPENDIX II. TIME VARYING DEFICIT OF DO**

The equation for dissolved oxygen deficit in a flowing stream with a photosynthetic source of oxygen \( P_{mg}(t) \) is

\[ \frac{\partial D}{\partial t} + U \frac{\partial D}{\partial x} + K_D D = - P_{mg}(t) \] \hspace{1cm} (16)

It is understood that \( P_{mg}(t) \) is zero for \( x < 0 \) and independent of \( x \) for \( x > 0 \). Formally this can be expressed by multiplying \( P_{mg}(t) \) by a unit step function in \( x \), i.e., \( P_{mg}(t) \) \( u(x) \) when

\[
\begin{align*}
  u(x) &= 0 \quad x < 0 \\
        &= 1 \quad x > 0
\end{align*}
\] \hspace{1cm} (17)

Let the initial condition at \( t = 0 \) be

\[ D(x, 0) = 0 \quad x > 0 \] \hspace{1cm} (18)

and the boundary condition at \( x = 0 \) be an arbitrary function \( D_B(t) \), i.e.,

\[ D(0, t) = D_B(t) \quad t > 0 \] \hspace{1cm} (19)

Define the Laplace transform of \( D(x, t) \) with respect to \( t \) as

\[ \mathcal{L}(D(x, t)) = \int_0^\infty e^{-st} D(x, t) dt \] \hspace{1cm} (20)

The following equation can be used (4):

\[ \int_0^\infty e^{-st} \frac{\partial D(x, t)}{\partial t} dt = sD(x, s) - D(x, 0) \] \hspace{1cm} (21)

under a certain assumption about the nature of \( D \).

Applying this transform to Eq. 16 yields

\[ sD(x, s) + U \frac{\partial D}{\partial x} + K_D D = - \mathcal{L}(P_{mg}(t)) \] \hspace{1cm} (22)

in which the initial condition \( D(x, 0) = 0 \) has been incorporated. Eq. 22 is a differential equation in \( x \) only and has as its general solution

\[ D(x, s) = A(s) e^{-[(K_a + s)/U]x} + \frac{P_{mg}(s)}{s + K_a} \] \hspace{1cm} (23)

To evaluate the arbitrary function \( A(s) \) the boundary condition, Eq. 19, is used. The Laplace transform of the boundary condition is

\[ D(0, s) = D_B(s) \] \hspace{1cm} (24)

Evaluating Eq. 23 at \( x = 0 \), and using Eq. 24 yields

\[ D_B(s) = A(s) - \frac{P_{mg}(s)}{s + K_a} \] \hspace{1cm} (25)

which when solved for \( A(s) \) yields

\[ A(s) = D_B(s) + \frac{P_{mg}(s)}{s + K_a} \] \hspace{1cm} (26)

Therefore, the solution becomes

\[ D(x, s) = \frac{P_{mg}(s)}{s + K_a} \left[ 1 - e^{-[(K_a + s)/U]x} \right] + \frac{D_B(s)e^{-[(K_a + s)/U]x}}{s + K_a} \] \hspace{1cm} (27)

In order to find the inverse transform of this equation, the form of \( g(t) \) is required. Because \( g(t) \) is the sum of cosines, it suffices to obtain the solution for \( g(t) = \cos(\omega t) \). The Laplace transform of \( g(t) \) is

\[ \mathcal{L}(g(t)) = \frac{s}{\omega^2 + s^2} \] \hspace{1cm} (28)

The inverse Laplace transform of the three terms in Eq. 27 are obtained from the following inverse transforms (14):

\[ L^{-1} \left[ e^{-at} \mathcal{L}(f)(s) \right] = f(t - b) u(t - b) \] \hspace{1cm} (29)

\[ L^{-1} \left[ \frac{as}{(\omega^2 + s^2)(a + b)} \right] = \frac{a}{\sqrt{\omega^2 + s^2}} \left[ \frac{b \cos(\omega t) + \omega \sin(\omega t) - b e^{-at}}{\omega^2 + s^2} \right] \] \hspace{1cm} (30)

\[ L^{-1} \left[ \frac{as}{\sqrt{\omega^2 + s^2}} \right] = \frac{a}{\sqrt{\omega^2 + s^2}} \left[ \frac{b \cos(\omega t) + \omega \sin(\omega t) - b e^{-at}}{\omega^2 + s^2} \right] \] \hspace{1cm} (31)
and 
\[ \sqrt{a^2 + \omega^2} \cos \left( \frac{\omega t}{\sqrt{a^2 + \omega^2}} \right) \] 

The inverse transforms corresponding to the three terms of the right-hand side of Eq. 27 for large \( t \) and for \( g(t) = e^{-\omega t} \) are 
\[ \frac{-P_m}{\sqrt{K_a^2 + \omega^2}} \cos \left( \frac{\omega t}{\sqrt{K_a^2 + \omega^2}} \right) \] 
\[ + \frac{P_m}{\sqrt{K_a^2 + \omega^2}} e^{-\omega t} \cos \left( \frac{\omega t - \frac{\omega}{U}}{\sqrt{K_a^2 + \omega^2}} \right) \] 
\[ + D_0 \left( 1 - \frac{\omega}{U} \right) e^{-K_a t/U} \] 

Using the Fourier series expansion for \( P_m g(t) \) (see Appendix I) in place of \( e^{-\omega t} \), the complete solution is 
\[ D(x, t) = D_0 \left( 1 - \frac{\omega}{U} \right) e^{-K_a t/U} \] 
\[ - P_m \frac{2}{\sqrt{K_a^2 + (2\pi/n)^2}} \cos \left( \frac{2\pi n (t - \frac{P}{2} - \frac{x}{U})}{K_a} \right) \] 
\[ + \sum_{n=1}^{\infty} \frac{b_n}{\sqrt{K_a^2 + (2\pi/n)^2}} \cos \left( \frac{2\pi n (t - \frac{P}{2} - \frac{x}{U})}{K_a} \right) \] 

in which the solution for the constant portion of \( P_m g(t) \), i.e., \( P_m = \frac{2P}{\pi} = \frac{2P}{n} \), the daily average value of \( P_m g(t) \), has been included.

APPENDIX III—REFERENCES

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APPENDIX IV—NOTATION

The following symbols are used in this paper:
\[ A = \text{cross-sectional area} \]
\( b_n \) = \( n \)th Fourier coefficient of photosynthetic oxygen source;

\( c_x \) = saturation value of dissolved oxygen;

\( c(x,t) \) = concentration of dissolved oxygen;

\( D(x,t) \) = dissolved oxygen deficit concentration;

\( D_0(t) \) = boundary condition of dissolved oxygen deficit;

\( D_L \) = diffusivity of dissolved oxygen;

\( H \) = river depth;

\( K_q \) = reseration coefficient;

\( K_d \) = deoxygenation coefficient of carbonaceous BOD;

\( K_R \) = deoxygenation coefficient of nitrogenous BOD;

\( L(s) \) = carbonaceous BOD concentration;

\( L_0 \) = \( L(0) \);

\( N_x \) = nitrogenous BOD concentration;

\( N_0 \) = \( N(0) \);

\( P(t) \) = photosynthetic oxygen source;

\( P_m \) = maximum rate of photosynthetic oxygen production;

\( P_{av} \) = average daily photosynthetic oxygen production;

\( p \) = duration in time of photosynthetic oxygen source;

\( Q \) = volumetric flow rate;

\( R \) = algal respiration sink of dissolved oxygen;

\( S(c,x,t) \) = total sources and sinks;

\( s \) = benthic respiration sink of dissolved oxygen;

\( s = \) LaPlace transform variable;

\( T \) = period of photosynthetic oxygen source = 1 day;

\( t \) = time;

\( t_1 \) = time at which photosynthetic source begins;

\( U \) = \( Q/A \) = stream velocity;

\( W \) = loading rate, in pounds per day BOD;

\( x \) = distance downstream;

\( x/U \) = travel time from \( x = 0 \); and

\( \omega_0 \) = radian frequency of the photosynthetic oxygen source = \( 2\pi/T \).